CHENNAI MATHEMATICAL INSTITUTE

Graduate Programme in Mathematics

Entrance Examination, 2011

Part A

State whether True or False and give brief reasons. Marks will be given only when reasons are provided. Answer any 10 questions in this part. All questions carry 5 marks.

1. There is a sequence of open intervals $I_n \subset \mathbb{R}$ such that $\bigcap_{n=1}^{\infty} I_n = [0,1]$.
2. The set S of real numbers of the form $\dfrac{m}{10^n}$ with $m, n \in \mathbb{Z}$ and $n \geq 0$ is a dense subset of $\mathbb{R}$.
3. There is a continuous bijection from $\mathbb{R}^2 \to \mathbb{R}$.
4. There is a bijection between $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$.
5. If $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive real numbers with the first converging to zero, and the second diverging to ∞, then the sequence of complex numbers $c_n = a_n e^{ib_n}$ also converges to zero.
6. For any polynomial $f(x)$ with real coefficients and of degree 2011, there is a real number b such that $f(b) = f'(b)$.
7. If $f\colon [0,1] \to [-\pi, \pi]$ is a continuous bijection then it is a homeomorphism.
8. For any $n \geq 2$ there is an $n \times n$ matrix A with real entries such that $A^2 = A$ and trace $(A) = n + 1$.
9. There is 2×2 real matrix with characteristic polynomial $x^2 + 1$.
10. There is a field with 10 elements.
11. There are at least three non-isomorphic rings with 4 elements.
12. The group $(\mathbb{Q}, +)$ is a finitely generated abelian group.
13. $\mathbb{Q}(\sqrt{7})$ and $\mathbb{Q}(\sqrt{17})$ are isomorphic as fields.
14. A vector space of dimension ≥ 2 can be expressed as a union of two proper subspaces.
15. There is a bijective analytic function from the complex plane to the upper half-plane.
16. There is a non-constant bounded analytic function on $\mathbb{C} \setminus \{0\}$.

Part B

Answer any five questions. All questions carry 10 marks

1. (a) Consider the ring R of polynomials in n variables with integer coefficients. Prove that the polynomial $f(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$ has $2^{n+1} - 2$ non-constant polynomials in R dividing it.

   (b) Let $p_1, p_2, \ldots, p_n$ be distinct prime numbers. Then show that the number $N = p_1 p_2^2 p_3^3 \cdots p_n^n$ has $(n+1)!$ positive divisors.

2. Let $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 6)$. For every prime number p, show that $f(x) \equiv 0 \pmod{p}$ has a solution in $\mathbb{Z}$.

3. Let **S** denote the group of all those permutations of the English alphabet that fix the letters T,E,N,D,U,L,K,A and R. Other letters may or may not be fixed. Show that **S** has elements σ, τ of order 36 and 39 respectively, but does not have any element of order 37 or 38.

4. Show that there are at least two non-isomorphic groups of order 198. Show that in all those groups the number of elements of order 11 is the same.

5. Suppose f, g, h are functions from the set of positive real numbers into itself satisfying $f(x)g(y) = h(\sqrt{x^2 + y^2})$ for all $x, y \in (0, \infty)$. Show that the three functions $f(x)/g(x), g(x)/h(x)$, and $h(x)/f(x)$ are all constant.

6. Let $a, b > 0$.

   (a) Prove that $\lim_{n \to \infty} (a^n + b^n)^{1/n} = \max\{a, b\}$.

   (b) Define a sequence by $x_1 = a, x_2 = b$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n > 2$. Show that $\{x_n\}$ is a convergent sequence.

7. Let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function with the following property: *In the power series expansion around any $a \in \mathbb{C}$, given as $f(z) = \sum_{n=o}^{\infty} c_n(a)(z-a)^n$, the coefficient $c_n(a)$ is zero for some n (with n depending on a).* Show that $f(z)$ is in fact a polynomial.

8. (a) Show that in a Hausdorff topological space any compact set is closed.

   (b) If (X, d_1) and (Y, d_2) are two metric spaces that are homeomorphic then does completeness of (X, d_1) imply the completeness of (Y, d_2)? Give reasons for your answer.

9. Fix an integer $n > 1$. Show that there is a real $n \times n$ diagonal matrix D such that the condition $AD = DA$ is valid only for a diagonal matrix A.