INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2002

Program Integrated Ph.D

Entrance Paper Mathematical Sciences

Day & Date
SUNDAY 28TH APRIL 2002

Time
1.30 P.M. TO 4.30 P.M.

MATHEMATICAL SCIENCES

General Instructions

- The question paper is in two parts: Part A and Part B
- Part A carries 30 marks and Part B carries 70 marks.
- There is no negative marking.
- All answers must be written in the answer book and *not on the question paper*

Notations: The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Part A

Part A consists of 30 multiple choice questions each carrying mark.

Answer all questions from Part A.

- Four possible answers are provided for each question 'ick √) the correc answer against A, B, C or D on page 3 of the answer book.

If $(a, b) \neq (0, 0)$ then the real polynomial $x + ax + b$ must have ✓

A. only real zeros.

B. only non-real complex zeros.

C. a real zero.

D. a non-real complex zero.

For any integer $n \geq 3$, the value of $\frac{1}{n+1} + \frac{1}{n+2} +$ $\frac{1}{2n}$ is always

A

B.

C. n

D $\frac{2}{3n}$

3. Let ρ be a non-trivial relation on a set X If is ;ymmetric and antisymm then ρ is

A reflexive

B. transitive

C. an equivalence relation.

D. the diagonal relation (i.e $\rho y \Leftrightarrow x$ y)

_et $f: \mathbb{Z}$ $\mathbb{R}$ be defined by $f(x)$ x^3 $3x$ 1 Then f is

A not a function

B surjective (onto) function

an injective (one-to-one) function.

function but neither injectiv nor surjectiv

Suppose f $\mathbb{R}$ $\mathbb{R}$ is defined by .2001 'hen

A does not hav inverse over whole of $\mathbb{R}$.

B. has no inverse outside a finite open subset of $\mathbb{R}$.

C. has no inverse outside a finite closed subset of $\mathbb{R}$

D. has inverse over the whole of $\mathbb{R}$.

✓ 6. The set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40 The identity element of this group is

A. 5.

B. 15

C. 25

D. 35

Order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ is

A. 3.

B. 4.

C. 7.

D. 12.

✓ 8. Let $\mathbb{Z}_n$ be the additive group of integers modulo n The number of homomorphisms from $\mathbb{Z}_n$ to itself is

A. 0.

B. 1.

C. n.

D. n^2

✗ 9. The number of non-isomorphic abelian group(s) of order 15 is

A. 1.

B. 2

C. 3

D. 4.

10. Let R be a commutative ring. An element $x \in R$ is said to be nilpotent if $x^n = 0$ for some positive integer n. If x and y in R are such that x and $x + y$ are nilpotents then y must be

A. the additive identity of R.

B. the multiplicative identity of R

C. x^m, for some integer m.

D. nilpotent.

1 The characteristic polynomial of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is

A. $x(x^2+1)$.

B. $x(x-1)^2$

C. $x(x+1)^2$.

D. $x(x^2-1)$

12. Let $v = (1,1)$ and $w = (1,-1) \in \mathbb{R}^2$ Then a vector $u = (a,b) \in \mathbb{R}^2$ is in the $\mathbb{R}$-linear span of v and w

A. only when $a = b$.

B. always.

C. for exactly one value of (a,b).

D. for at most finitely many values of (a,b)

13. The dimension of the vector space $\{(x,y,z,w) \in \mathbb{R}^4 \quad w, x+z = y = z+w\}$ is

A 0.

B.

C. 2

D.

14. Let A be a 3×3 real matrix. Suppose $A^4 = 0$. Then A has

A. exactly two distinct real eigenvalues.

B. exactly one non-zero real eigenvalue.

C. exactly 3 distinct real eigenvalues.

D. no non-zero real eigenvalue.

15. Let a, b, c, d be real numbers and let $f : \mathbb{C} \to \mathbb{C}$ be the map defined by $f(x+iy)$ $(ax+by)+i(cx+dy)$. Then f is linear over $\mathbb{C}$ if and only if

A. $a = d$ and $b = c$.

B. $a = d$ and $b = -c$

C. $a = -d$ and $b = c$

D. a d and b c

16. Let C_1 and C_2 be two distinct ellipses in the plane. If C_1 and C_2 have a common tangent at a common point P then the number of distinct common points of C_1 and C_2 must be

A. 1.

B. 1 or 2.

C. 1, 2 or 3.

D. 1, 2, 3 or 4.

17. Let P $\{(x, y, z) \in \mathbb{R}^3 : x = 0\}$, $Q = \{(x, y, z) \in \mathbb{R}^3 : y$ $0\}$, $R = \{(x, y, z) \in \mathbb{R}^3$ $x + y$ $1\}$ be three planes in $\mathbb{R}^3$. Let $l = P \cap R$ and m $Q \cap R$. Then l and m

A. are two skew lines.

B. are two parallel lines.

C. intersect at the origin.

D. are perpendicular to each other

18. Let S be unit sphere with center $(0, 0, 1$ in $\mathbb{R}^3$ and P be the plane z Then the equation of $S \cap P$ is

A. $x^2 + y^2$ $\frac{3}{4}$, $z = \frac{1}{2}$

B. $x^2 + y^2$ 1, $z = \frac{1}{2}$

C. $x^2 + y^2$ $2x = 1$, z $\frac{1}{2}$

D. $x^2 + y^2$ $2y = \frac{3}{4}$, z $\frac{1}{2}$

19. The three lines $ax + a^2y$ $bx + b^2y$ $cx + c^2y$ in $\mathbb{R}^2$ are concurrent if and only if

A. $a = b = c$

B. two of a, b, c are equal

C. a, b, c are all distinct.

D. $a = c^2$ and $b = c^3$.

20. The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x)$ $\max\{1 - |x$ $0\}$ is differentiable

A. at all points.

B. at all except one point.

C. at all except three points.

D. nowhere.

/21 Let $f:[0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1)$. If f is differentiable on $(0,1)$ and the derivative f' is continuous on $(0,1)$ then f' is

A. strictly positive in $(0,1)$.

B. strictly negative in $(0,1)$.

C. identically zero in $(0,1)$.

D. zero at some point in $(0,1)$

22. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for

A. all $p > 0$.

B. for only $p = 1$.

C. for all $p > 1$.

D. for all integer values of p.

23. Let $\mathbf{V}$ $\mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{V}(x_1, x_2, x_3) := (x_1^2 + x_2^2, x_1x_2 + x_2x_3, x_2^2 + x_1x_3)$$

The divergence of $\mathbf{V}$ is

A. $4x_1 + x_3$.

B. 0.

C. $x_1^2 + x_2^2 + 2x_1x_3$

D. $(2x_1, x_1 + x_3, x_1$

/ 24 A unit normal vector to the curve $\mathbf{C} := \{(x, x^2) \; x \in \mathbb{R}\}$ in the plane $\mathbb{R}^2$ at the point $(0,0)$ is given by

A. $(0, -1)$.

B. $(-1, 0)$.

C. $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

D. $(1, 0)$.

25. The number of zeros of the function $f(x) = \sin x \cos x$ in $(0, n\pi)$ is

A. $n + 1$.

B. $2n - 1$

C. $2n$.

D. $2n + 1$

26. The function $f: [-1, 1] \to \mathbb{R}$ defined by $f(x) = 1 - x^2$ has

A. no local maxima or minima in $(-1, 1)$

B. has exactly one local maximum and two local minima in $(-1, 1)$

C. has exactly one local maximum in $(-1, 1)$.

D. has exactly one local minimum in $(-1, 1)$.

27. If $f(x, y) = x^7 + 100x^5y^2 + 200xy^6 + 10y^7$ then $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}$ is =

A. $42x^7 + 4200x^5y^2 + 8400xy^6 + 420y^7$.

B. $42x^7 + 500x^5y^2 + 200xy^6 + 10y^7$.

C. $42x^7 + 1000x^5y^2 + 1200xy^6 + 420y^7$

D. $7x^7 + 700x^5y^2 + 1400xy^6 + 70y^7$.

28. A solution of the differential equation $\frac{dy}{dx} = y + 1$ is

A. $y = e^x - 1$.

B. $y = e^x + 1$.

C. $y = e^x + x$

D. $y = e^{x-1}$.

29. The differential equation $\frac{d^2y}{d^2x} + 4\frac{dy}{dx} + 4y = 0$ has general solution of the form

A. $A\cos 2x + B\sin 2x$

B. $Ae^{-2x} + Bxe^{-2x}$.

C. $Ae^{2x} + Bxe^{2x}$.

D. $Ae^{2x} + Be^{-2x}$.

30. The iteration $x_{n+1} = x_n^2 - 2$, $x_n \geq 0$ for $n \geq 1$ will converge to the solution $x = 2$ of the equation $x^2 - x - 2 = 0$ if and only if x_1 is

A. close to 2 from the left.

B. close to 2 from the right.

C. equal to 2.

D. equal to $\sqrt{2}$.

Part B

- Part B comprises 24 questions. Each question carries 5 marks
- Answer **any 14** full questions only.
- Only the **first 14 answered** questions will be evaluated
- Answer should be to the point.

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1. Let a, b be real numbers and let $f, g: \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$\begin{aligned} f(x &= ax + b \text{ and} \\ g(x &= x^2, \end{aligned}$$

respectively. Show that $f \circ g \quad g \circ f$ if and only if $(a, b) = (0,0)$, $(0,1)$ or $(1,0)$

2. For an integer $n \geq 4$, compute the $n \times n$ determinant

$$\begin{matrix} & 2 & 3 & & n \\ 1 & 2^3 & 3^3 & & n^3 \\ & & \cdot & & \\ 1 & 2^{2n-1} & 3^{2n-1} & & n^{2n} \end{matrix}$$

3. For all $n \in \mathbb{N}$ and for all positive real numbers x, y, show that

$$\left(1 + \frac{x}{y}\right) + \left(1 + \frac{y}{x}\right) \geq 2^{n+1}$$

4. Let ρ be a relation on a non-empty set X. For $Y \subseteq X$, let

$$\mathrm{N}(Y) := \{x \in X \mid \text{ there exists } y \in Y \text{ such that } y\rho x\}$$

Show that ρ is reflexive if and only if $Y \subseteq \mathrm{N}(Y$ for all $Y \subseteq X$

5. Let $f: \mathbb{Z} \to \mathbb{R}$ be a function. If $f(n) \quad f(n+11) = f(n+18)$ for all $n \in \mathbb{Z}$ then show that f is a constant function.

6. Let '+' and '·' be the operations on the set $C[0,1]$ of continuous real valued functions on $[0,1]$, defined by

$$\begin{aligned} (f+g)(x) &\quad f(x) + g(x) \\ (f \cdot g)(x) &= f(x) \cdot g(x), \end{aligned}$$

for all $x \in [0,1]$. Show the following.
(a) $(C[0,1], +, \cdot)$ is a ring. [3 marks]
(b) $(C[0,1], +, \cdot)$ has a divisor of zero. [2 marks]

7. If

$$x_2 + x_3$$

$$\begin{aligned} x_{98} + x_{99} + x_{100} & \quad 0, \\ x_{99} + x_{100} + x_1 & \quad 0, \\ x_{100} + x_1 + x_2 & \quad 0, \end{aligned}$$

then show that $x_1 \quad x_2 \qquad = x_{99} = x_{100} = 0$.

8. Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, y) := (x, x + y, y)$.
(a) Show that f is linear. [2 marks]
(b) Find the range and the kernel of f. [3 marks]

9. Let P, Q and R be three non-collinear points in the plane. Show that every point X in the plane can be uniquely written as $X = a_1 P + a_2 Q + a_3 R$, where a_1, a_2, a_3 are real numbers with $a_1 + a_2 + a_3 = 1$.

10. Find the volume of the largest (right circular) cone that can be inscribed in a sph of radius $R > 0$.

Let $f, g: \mathbb{R} \to \mathbb{R}$ be two continuous functions. Show that the function $h: \mathbb{R} \to \mathbb{R}$ defined by

$$h(x \qquad \max\{f(x \quad g(x)\} \quad \text{for } x \in \mathbb{R}.$$

is continuous.

12 Let $f: [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b] \quad \mathbb{R}$ with $f(a) = f(b)$. Show that there exists $c \in \left[a, \frac{a+b}{2}\right]$ such that $f\left(c + \frac{b-a}{2}\right) = f($

1 Let $f: [0, 1] \to \mathbb{R}$ be a continuous function. Suppose hat $f(r) = r^3 + 99r + 100$ for every rational number $r \in [0, 1]$. Prove that $f(x) = \quad {}^3 + 99x + 100$ for all $x \in [0, 1]$.

14. Does the series $\sum_{n=1}^{\infty} \frac{(n!)^2 5^n}{(2n)!}$ converge? Justify your answer.

15. If a sequence a_n, $n \in \mathbb{N}$ of real numbers is monotone decreasing and the series $\sum_{n=0}^{\infty} a_n$ is convergent, then show that the sequen $na_n \quad n \in \mathbb{N}$ converges to 0

16. Show that the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) := x^3 + 2x +$ is strictly increasing and compute the derivative $(f^{-1})'(1)$ of the inverse function f^{-1} at the point $1 = f(-1)$.

17. Let $f: \mathbb{R} \to \mathbb{R}$ be a function which is 3-times differentiable in a neighbourhood of 0 and $f(0) = 0$. Show that the function $g: \mathbb{R} \to \mathbb{R}$, defined by

$$g(x \qquad \begin{cases} f(x) & \text{if } x \neq 0 \\ f'(0) & \text{if } x \end{cases}$$

is differentiable at 0 and $g'(0) = \frac{1}{2}f''(0)$.

18. For $n \in \mathbb{N}$, let

$$a_n := \int_0^{\pi/2} \sin^n t \, dt.$$

Show the following.

(a) $(n+1)a_{n+1} = na_{n-1}$ for $n \geq 1$ [2 marks]

(b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ [3 marks]

19. Let $f: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by

$$f(v, w) := v \times w \quad (\text{the vector product of } v \text{ and } w).$$

Show that f is surjective (onto)

20. Let $f: (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) := x^x$. Find local maxima and minima of f.

21 Find out all the local maxima, local minima and points of inflection of the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x^5 - 5x^3 + 15$.

22. Show that any solution y of the differential equation

$$\frac{dy}{dx} = \sin y$$

on an interval $[0, a]$ statisfies

$$|y(x) - y(0)| \leq x \quad \text{for all} \quad x \in [0, a]$$

23. Describe the Euler numerical scheme and the Runge-Kutta method of order 2 for solving the differential equation

$$\begin{aligned} \frac{dy}{dx} &= f(y), \quad x \in \mathbb{R} \\ y(0) &= y_0, \end{aligned}$$

where $f: \mathbb{R} \to \mathbb{R}$ is differentiable and the derivative f' is continuous on $\mathbb{R}$. Explain also why the Runge-Kutta method is preferred to the Euler method.

24. Compute the area of the region

$$R := \left\{(x, y) \in \mathbb{R}^2 \quad \max\{|x|, |y|\} \leq 1 \quad 4xy \leq 1\right\}$$