INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2003

Program Integrated Ph.D

Entrance Paper Mathematical Sciences

Day & Date
SUNDAY 27th APRIL 2003

Time
1.30 P.M. TO 4.30 P.M.

General Instructions

- This question paper has two parts : Part A and Part B .
- Part A Carries 30 marks and Part B carries 70 marks.
- All answers must be written in the answer-book and **not on the question paper**.

Notation

The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Integrated Ph. D. / Mathematical Sciences

Part A

- Part A consists of 30 multiple choice questions, each carrying 1 mark.
- Answer **all** questions.
- Four possible answers are provided for each question. Tick ($\surd$) against correct answer, namely, A, B, C or D on the Page 3 of the answer book.

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1. The number of reflexive relations on the set $\{1, 2, \ldots, n\}$ is

(A) $2^{n(n-2)}$.
(B) $2^{n(n-1)}$.
(C) 2^{n^2}.
(D) $2^{n(n+1)}$.

2. For any two natural numbers n and k, the number of all k-tuples $(a_1, \ldots, a_k) \in \mathbb{N}^k$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$ is

(A) $\binom{n}{k}$.
(B) $\binom{n+k}{k}$.
(C) $\binom{n+k-1}{k}$.
(D) $\binom{n+k+1}{k}$.

3. The probability that a hand of 5 playing cards contains at least two aces is

(A) $\dfrac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}}$.

(B) $\dfrac{\binom{4}{2}+\binom{48}{3}}{\binom{52}{5}}$.

(C) $\dfrac{\binom{4}{2}\cdot\left[\binom{48}{3}+\binom{48}{2}+\binom{48}{1}\right]}{\binom{52}{5}}$.

(D) $\dfrac{\binom{4}{2}\cdot\binom{48}{3}+\binom{4}{3}\cdot\binom{48}{2}+\binom{4}{4}\cdot\binom{48}{1}}{\binom{52}{5}}$.

4. Let a, b, c, d be rational numbers with $ad - bc \neq 0$. Then the function $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ defined by $f(x) := \frac{ax+b}{cx+d}$ is

(A) onto but not one-one.
(B) one-one but not onto.
(C) neither one-one nor onto.
(D) both one-one and onto.

5. The supremum of the set $\left\{\frac{n^2}{2^n} \quad n \in \mathbb{N}\right\}$

(A) is $\frac{9}{8}$.
(B) is 1
(C) is 0.
(D) does not exist.

6. Let $n \in \mathbb{N}$. Then the complex number $\left(\frac{1+i}{\sqrt{2}}\right)^n$ is purely imaginary if and only if

(A) $n \equiv 0 \pmod 4$.
(B) $n \equiv 1 \pmod 4$.
(C) $n \equiv 2 \pmod 4$.
(D) $n \equiv 3 \pmod 4$.

(For $a, b, m \in \mathbb{Z}, m > 1$, $a \equiv b \pmod m$ means that m divides $a - b$

7. The equation $\frac{1}{1+x^2} = \sqrt{x}$, $x \geq 0$ has

(A) no real solution.
(B) exactly one real solution.
(C) exactly 3 real solutions.
exactly 5 real solutions.

8. Let $p(X) := X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n$ be a real polynomial of degree $n \geq 1$. If n is even and a_n is negative, then

(A) f has at least one positive and one negative real zero.
(B) all real zeros of f are positive.
(C) all real zeros of f are negative.
f has no real zeros.

9. The points of intersection of the two plane curves defined by the equations $y^2 = a^2$ and $(y - bx)^2 = c^2$, $a, b, c \in \mathbb{R}$, $b \neq 0$ are vertices of

(A) an equilateral triangle.
(B) a square.
(C) a reactangle.
a paralleogram.

10. Let V be the $\mathbb{R}$-vector space of all functions $\mathbb{R} \to \mathbb{R}$ and let W be the $\mathbb{R}$-subspace of V generated by the functions $\sin t$, $\sin(t+1)$, $\sin(t+2)$. Then the dimension of W is

(A) 0
(B) 1
(C) 2
(D) 3

11. Let $n \in \mathbb{N}$, $n \geq 3$ and let $x_k := (kn+1, kn+2, \ldots, kn+n)$, $k = 0, 1, \ldots, n-1$. Then the maximal linearly independent subsequence of the sequence $x_0, x_1, \ldots, x_{n-1} \in \mathbb{R}^n$ has the length

(A) 1.
(B) 2.
(C) $n-1$
(D) n

12. For real numbers a, b, c, the following linear system of equations

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ ax & + & by & + & cz & = & 1 \\ a^2x & + & b^2y & + & c^2z & = & 1 \end{array}$$

has a unique solution if and only if

(A) $b = c$ and $b \neq a$.
(B) $a = b$ and $a \neq c$.
(C) $a = c$ and $a \neq b$.
(D) $a \neq b$, $b \neq c$ and $a \neq c$

13. For $r, s \in \mathbb{N}$, the signature of the permutation

$$\sigma := \begin{pmatrix} 1 & 2 & & r-1 & r & r+1 & r+2 & & r+s \\ s+1 & s+2 & & s+r-1 & s+r & 1 & 2 & & s \end{pmatrix}$$

is

(A) $(-1)^{rs}$.
(B) $(-1)^{r+s}$
(C) $(-1)^{r}$.
(D) $(-1)^{s}$.

14. The number of subgroups in the cyclic group of order 360 is

(A) 6.
(B) 8.
(C) 12.
(D) 24.

15. Let m be an odd integer > 6. Then the multiplicative inverse of 2 in the ring $(\mathbb{Z}_m, +_m, \cdot_m)$ (where $+_m$ and $\cdot_m$ denote the addition and multiplication modulo m respectively.)

(A) does not exist.
(B) is $\frac{m-1}{2}$.
(C) is $\frac{m+1}{2}$.
(D) is $m-2$.

16. The power set $\mathfrak{P}(X)$ of a set X with the binary operations symmetric difference[1] Δ and intersection $\cap$ form a ring (the symmetric difference is the addition and the intersection is the multiplication) called the power set ring of the set X. If the set X has at least 3 elements, then in the power set ring $(\mathfrak{P}(X), \Delta, \cap)$ of X, every element is

(A) a unit.
(B) idempotent.
(C) nilpotent.
(D) a non-zero divisor.

17. The polynomial $f(X) := X^3 + aX + 1$ in $\mathbb{Z}_3[X]$ is

(A) irreducible in $\mathbb{Z}_3[X]$ if and only if $a = -1$.
(B) irreducible in $\mathbb{Z}_3[X]$ if and only if $a = 0$.
(C) irreducible in $\mathbb{Z}_3[X]$ if and only if $a = 1$.
(D) always reducible in $\mathbb{Z}_3[X]$.

18. Let x be a rational number which is not an integer. Then the sequence $a_n(x) := nx - [nx]$, $n \in \mathbb{N}$, (for any real number y, the bracket $[y]$ denote the largest integer $\leq y$) has

(A) infinitely many limit points.
(B) at least 2, but finitely many limit points.
(C) exactly one limit point.
(D) no limit point.

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[1]For $A, B \in \mathfrak{P}(X)$, the subset $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is called the symmetric difference of A and B

19. The sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots,$

(A) is a divergent sequence.

(B) is convergent and its limit is $\leq \sqrt{2}$.

(C) is convergent and its limit is $\geq 3/2$.

(D) is convergent and its limit is ≥ 4.

20. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) := \dfrac{e^x}{x^x}$. Then the limit $\lim_{x\to\infty} f(x)$

(A) does not exist.

(B) exists and is 0.

(C) exists and is 1.

(D) exists and is e.

21. The series $\sum_{n=1}^{\infty} \dfrac{(-1)^n n!}{n^n}$ is

(A) absolutely convergent.

(B) conditionally convergent

(C) oscillatory.

(D) divergent.

22. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x, & \text{if } x \in \mathbb{Q},\ x > 0, \\ -x, & \text{if } x \in \mathbb{Q},\ x \leq 0, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then f is

(A) neither left continuous nor right continuous at 0.

(B) left continuous but not right continuous at 0.

(C) right continuous but not left continuous at 0.

(D) continuous at 0.

23. If tangent at the origin to the curve defined by the equation $y = ax + bx^2 + cx^3$ passes through the point (a, b), then

(A) $b = -a^2$.

(B) $b = a^2$.

(C) $b = -a$.

(D) $b = a$.

24. Let $x(t)$ and $y(t)$ be two non-constant differentiable real valued functions on $\mathbb{R}$ such that

$$\frac{d\,x(t)}{dt} = -y(t) \quad \text{and} \quad \frac{d\,y(t)}{dt} = x(t)$$

Then the plane curve $t \mapsto (x(t), y(t))$ is

(A) a constant curve.
(B) a straight line.
(C) a circle.
(D) a parabola.

25. The derivative of the function $\mathbb{R}^+ \to \mathbb{R}$, $x \mapsto x^x$ is

(A) $(\ln x + 1)\, x^x$
(B) $(\ln x + x)\, x^x$
(C) $\left(\ln x + \frac{1}{x}\right) x^x$
(D) $x \cdot x^{x-1}$.

26. Let α, β be two real numbers and $\beta > 0$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ x^\alpha \sin(1/x^\beta), & \text{if } x > 0. \end{cases}$$

is differentiable at 0 if and only if

(A) $\alpha = \beta$
(B) $\alpha > \beta$
(C) $\alpha < \beta$
(D) $\alpha > 1$

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at $a \in \mathbb{R}$ and $f(a) \neq 0$ Then the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := |f(x)|$ is

(A) differentiable at a and $g'(a) = f'(a)$.
(B) differentiable at a and $g'(a) = -f'(a)$.
(C) differentiable at a and $g'(a) = \text{sign}(f(a)) \cdot f'(a)$
(D) not differentiable at a.

28. The function $f(x) := ax^2 + bx + c$, $a, b, c \in \mathbb{R}$, $a \neq 0$ with $\lim_{x \to \infty} f(x) = \infty$ has

(A) a unique point of minimum in $\mathbb{R}$.
(B) a unique point of maximum in $\mathbb{R}$.
(C) exactly two points of minimum in $\mathbb{R}$.
(D) exactly two points of maximum in $\mathbb{R}$

29. Let $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ be the vector field defined by

$$F(x) := \frac{x}{\|x\|},$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$ and $\|x\| := \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then the divergence $\operatorname{div} F(x)$ of $F(x)$ is

(A) $\|x\|$.
(B) $1/\|x\|$.
(C) $2 \cdot \|x\|$.
(D) $2/\|x\|$.

30. For a partial differentiable vector field $v = (v_1, v_2, v_3) : U \to \mathbb{R}^3$ defined on an open subset $U \subseteq \mathbb{R}^3$, the vector product $\nabla \times v$ of ∇ and v is called the rotation field of v, where $\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. For a two times continuously partially differentiable function $f : U \to \mathbb{R}$, the rotation field of the gradient field $\operatorname{grad} f$ of f is

(A) $\operatorname{grad} f$.
(B) $2 \cdot \operatorname{grad} f$
(C) $(0, 0, 0)$.
(D) $(1, 1, 1)$.

Part B

- Part B consists of 24 questions, each carrying 5 marks.
- Answer **any 14** questions.
- Only the **first 14 answered** questions will be evaluated.

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1. For any natural number $n \geq 1$ prove the formula $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

2. For every real number $b > 1$, show that there exists a nautral number n_0 such that $b^n > n$ for all natural numbers $n \in \mathbb{N}$ with $n \geq n_0$.

3. Let $\leq$ denote the product order on $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, i.e. for two tuples (x_1, x_2), $(y_1, y_2) \in \mathbb{N}^2$, $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Show that every subset X of $(\mathbb{N}^2, \leq)$ has only finitely many minimal elements.

   (**Hint**: One may assume that if $x \in X$ and $x \leq y$, $y \in \mathbb{N}^2$, then $y \in X$.)

4. Let A be an uncountable subset of the set of all positive real numbers. For every real number r, show that there are finitely many distinct real numbers $a_1 \ldots, a_n \in A$ such that
$$a_1 + \cdots + a_n \geq r .$$

5. Let $a_1, \ldots a_n$ be distinct real numbers and let
$$F(x) := \frac{1}{x - a_1} + \frac{1}{x - a_2} + \cdots + \frac{1}{x - a_n}$$
   For any real number c, show that $F(x) = c$ has exactly $n - 1$ or n real solutions according as $c = 0$ or $c \neq 0$.

6. Let $n \geq 1$ and let $\mathbf{A}$ be a $n \times n$ real matrix of rank $n - 1$. Then show that the adjoint matrix $\operatorname{Adj}\mathbf{A}$ of $\mathbf{A}$ has rank 1.

7. Let $n \geq 1$ and let $\mathbf{A}$ be a $n \times n$ matrix with integer entries and let $a \in \mathbb{Q} \setminus \mathbb{Z}$. Show that the matrix $a\mathbf{I}_n + \mathbf{A}$ is invertible.

8. For every divisor d of $24 = 4!$, find the number $\alpha(d)$ of elements of order d in the permutation group $\mathfrak{S}_4$ on 4 symbols.

9. Let G be a group, e be the identity element in G and let $x \in G$ be an element of order 2. Show that $H := \{e, x\}$ is a subgroup of G. Further, show that H is normal if and only if x belongs to the center $Z(G) := \{y \in G \mid yz = zy \text{ for all } z \in G\}$.

10. Let a and b be real numbers and let $(a_n)_{n \in \mathbb{N}}$ be the sequence recursively defined by
$$a_0 := a, \; a_1 := b, \; a_n := \frac{1}{2}(a_{n-1} + a_{n-2}) \; \text{ for } n \geq 2 .$$
    Is the sequence $(a_n)_{n \in \mathbb{N}}$ convergent ? If the answer is yes, then find its limit.

    (**Hint**: Note that $a_{k+1} - a_k = -\frac{1}{2}(a_k - a_{k-1})$ for all $k \geq 1$.)

11. Let $h_n := \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}$, $n \geq 1$. Show that the series $\sum_{n=1}^{\infty} \frac{h_n}{2^n}$ is convergent and that

$$\sum_{n=1}^{\infty} \frac{h_n}{2^n} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n 2^n} .$$

12. Let $a \in \mathbb{R}$, $a > 0$ and let the sequences $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ are defined recursively by

$$x_0 := a, \quad x_{n+1} := \sqrt{x_n}, \quad y_n := 2^n (x_n - 1) \quad \text{for all } n \in \mathbb{N} .$$

Find $\lim_{n \to \infty} y_n$

13. Show that the series $\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2}\right)$ is convergent and find its sum.

(**Hint** : First prove the formula $\prod_{n=1}^{N} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2} \left(1 + \frac{1}{N}\right)$.)

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that f must be a multiplication by a fixed real number a. i.e. there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$. (**Hint** : First prove that $f(x) = f(1) \cdot x$ for all $x \in \mathbb{Q}$.)

15. For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined by $f_n(x) := \dfrac{nx}{1 + |nx|}$. Show that all the functions f_n, $n \in \mathbb{N}$ are continuous. For which real numbers x, is the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) := \lim_{n \to \infty} f_n(x)$ defined ? and for which x is it continuous ?

16. A function $f : \mathbb{R} \to \mathbb{R}$ is called e v e n if $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and is called o d d if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Show that,

   (a) The derivative of a differentiable even (respectively odd) function is odd (respectively even). [3 marks]

   (b) The polynomial function $f(x) := a_0 + a_1 x + \cdots + a_n x^n$, $a_0, \ldots, a_n \in \mathbb{R}$, is even (respectively odd) if and only if $a_k = 0$ for all odd (respectively even) indices k. [2 marks]

17. Let $\tanh : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\tanh(x) := \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Show that

   (a) $\tanh$ is strictly monotone increasing. [1 mark]

   (b) $\tanh$ maps $\mathbb{R}$ bijectively onto the open interval $(-1, 1)$. [$1\frac{1}{2}$ marks]

   (c) The inverse function $\tanh^{-1} : (-1, 1) \to \mathbb{R}$ is differentiable. [$1\frac{1}{2}$ marks]

   (d) Find the derivative of $\tanh^{-1}$. [1 mark]

18. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ e^{-1/x^2}, & \text{if } x > 0. \end{cases}$$

Show that f is 3-times continuously differentiable and compute the k-derivative $f^{(k)}$ of f for all $k = 1, 2, 3$.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. For any two real numbers a, b with $a < b$, show that there exists a real number $c \in (a, b)$ such that

$$\left| \frac{f(b) - f(a)}{b - a} - f'(a) \right| \leq |f'(c) - f'(a)|.$$

(**Hint :** Use mean value theorem.)

20. Let $n \geq 1$ be a natural number and let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x)$ $x^n e^{-x}$. Determine the maxima and minima of the function f.

21. Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions on the closed interval $[a, b] \subseteq \mathbb{R}$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Show that there exists a real number $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

22. Let t be a positive real number. Compute the area bounded by the hyperbola $y = \sqrt{x^2 - 1}$ and the two lines $y = (\tanh t) \cdot x$, $y = -(\tanh t) \cdot x$ passing through the points $(\cosh t, \sinh t)$, $(\cosh t, -\sinh t)$ respectively.

(**Hint :** Use the formula $\int_a^b \sqrt{x^2 - 1}\, dx = \frac{1}{2}\left[-\cosh^{-1}(x) + x\sqrt{x^2 - 1}\right]_a^b$.)

23. Show that the function $F : \left(\mathbb{R}^3 \setminus \{0\}\right) \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(x, t) := \frac{\cos\left(\|x\| - ct\right)}{\|x\|}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$, $t \in \mathbb{R}$ and $\|x\| := \sqrt{x_1^2 + x_2^2 + x_3^2}$ is a solution of the differential equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right) F(x, t) = 0$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator in dimension 3

24. Let a be a positive real number and let $f, g : (-a, a) \to \mathbb{R}$ be two continuous functions. Suppose that f is an odd function and g is an even function, i.e.

$$f(-x) = -f(x), \quad \text{and} \quad g(-x) = g(x) \quad \text{for all } x \in (-a, a).$$

Show that the differential equation $y'' + f(x) \cdot y' + g(x) \cdot y = 0$ has two linearly independent solutions one of which is even and the other is odd.