INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2004

Program Integrated Ph.D

Entrance Paper : Mathematical Sciences
Paper Code : MS

Day & Date
SUNDAY 25TH APRIL 2004

Time
1.30 P.M. TO 4.30 P.M.

Integrated Ph.D./Mathematical Sciences

General Instructions

(1) The question paper consists of two parts, Part A and Part B.
(2) Answers to Part A are to be marked in the OMR sheet provided.
(3) For each question darken the appropriate bubble to indicate your answer.
(4) Use only HB pencils for bubbling answers.
(5) Mark only one bubble per question. If you mark more than one bubble, the question be evaluated as incorrect.
(6) If you wish to change your answer, please erase the existing mark completely before marl the other bubble.
(7) Answers to Part B are to be written in the separate answer book provided.
(8) Candidates are asked to fill in the required fields on the sheet attached to the answer bc

Notation

The set of natural numbers, integers, rational numbers, real numbers, positive real number: complex numbers are denoted by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{C}$ respectively.

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Part A

- Part A contains 15 multiple choice questions.
- You will get 2 marks for each correct answer and -0.5 mark for each wrong answer.
- Four possible answers are provided for each question and only one of these is correct

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(1) Define a relation ρ on the set of positive integers $\mathbb{Z}^+$ by $x\rho y$ if and only if g.c.d. of x and y is bigger than 1. Then the relation ρ is

(A) reflexive and symmetric but not transitive.

(B) symmetric and transitive but not reflexive.

(C) symmetric but neither reflexive nor transitive.

(D) an equivalence relation.

(2) Let f be a polynomial of degree n, say $f(x) = \sum_{k=0}^{n} c_k x^k$ such that the first and last coefficients c_0 and c_n have opposite signs. Then

(A) $f(x) = 0$ for at least one positive x.

(B) $f(x) = 0$ for at least one negative x.

(C) $f(x) = 0$ for at least one positive x and for at least one negative x.

(D) f need not vanish anywhere.

3) Let a, b and c be arbitrary real numbers. Let A be the matrix

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Let I be the 3×3 identity matrix Then

(A) $A^2 - 3A + 3I = A^{-1}$.

(B) $A^2 + 3A + 3I = A^{-1}$.

(C) $A^2 + A + I = A^{-1}$.

(D) A is not invertible.

(4) Let A be the matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ where $a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ and $c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$ are three mutually orthogonal unit vectors in $\mathbb{R}^3$. Let A^t denote the transpose of A. Then

(A) $A = A^{-1}$

(B) $A^2 = A$.

(C) $A^t = A$.

(D) $A^t = A^{-1}$

(5) Let (x_1, y_1) and (x_2, y_2) be two points on the parabola $y^2 = 2mx$ with $x_1 \neq x_2$. If
and L_2 are the tangents of the parabola at these two points respectively, then the poin
intersection of L_1 and L_2 has coordinates

(A) $\left(\frac{y_1 y_2}{2m}, \frac{y_1+y_2}{2}\right)$

(B) $\left(\frac{x_1 y_2 + x_2 y_1}{y_1+y_2}, \frac{m(x_1+x_2)}{y_1+y_2}\right)$

(C) $\left(-\frac{m}{2}, 0\right)$.

(D) $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

(6) Let T be a tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$. If d_1 and d_2 are t
distances from the foci of the ellipse to T, then the product $d_1 d_2$ is

(A) equal to a^2.

(B) equal to b^2.

(C) at least $\frac{a^2+b^2}{2}$

(D) equal to ab.

(7) Let C be the circle with centre at the origin and radius $a > 0$. If P is a moving poi
in the plane such that its distance from the nearest point on the circle is the same as i
distance from the x-axis, then the locus of the point P is

(A) a parabola.

(B) a hyperbola.

(C) a pair of parabolas.

(D) a pair of straight lines.

(8) If $f \; [-1, 1] \to \mathbb{R}$ is a differentiable function with $f'(x) = 1 - |x|$ and $f(0) = 2004$ then

(A) $f(x) = 1002 + \frac{x - x|x|}{2}$

(B) $f(x) = 2003 + \frac{x}{2} - x|x|$

(C) $f(x) = 2004 + x - \frac{x|x|}{2}$.

(D) $f(x)$ can not be determined

(9) Let $x = \pi - 0.01814$ and $y = e + 0.40517$. If $z = 3.123456789101112$ then

(A) $z = x$.

(B) $z = y$.

(C) z is rational.

(D) z is irrational, $z \neq x$ and $z \neq y$.

(10) For a real number y, let $[y]$ denote the largest integer smaller than or equal to y. The value of the integral

$$\int_0^2 [x^2]dx$$

is equal to

(A) 1.

(B) $5 - \sqrt{2} \quad \sqrt{3}$.

(C) $3 - \sqrt{2}$.

(D) $8/3$.

(11 If f is an integrable function on the real line satisfying

$$\int_0^x tf(t)dt = \sin x - x\cos x - \frac{1}{2}x^2$$

for all real numbers x, then

(A) $f(x) = 1 - \cos x$.

(B) $f(x) = 1 + \sin x$.

(C) $f(x) = \sin x$.

(D) $f(x) = \sin x - 1$

12) Suppose $0 < p < 1$. Then

(A) $(\cos\theta)^p > \cos(p\,\theta)$ for all $\theta \in [0, \pi/2]$.

(B) $(\cos\theta)^p \leq \cos(p\,\theta)$ for all $\theta \in [0, \pi/2]$.

(C) $(\cos\theta)^p \leq p\,\cos(p\,\theta)$ for all $\theta \in [0, \pi/2]$.

(D) $(\cos\theta)^p$ and $\cos(p\,\theta)$ are not comparable in the interval $[0, \pi/2]$

(13) Let $A_1 > A_2 > \quad \cdot > A_k > 0$ be k real numbers. Then

$$\lim_{n\to\infty} (A_1^n + A_2^n + \quad + A_k^n)^{1/n}$$

is equal to

(A) $(A_1 + A_2 + \cdots A_k)/k$.

(B) 0.

(C) A_k.

(D) A_1.

(14) The value of the limit

$$\lim_{x\to 0} \frac{5^x - 3^x}{3^x - 2^x}$$

is

(A) $\log_e(10/9)$.

(B) $\log_{\frac{3}{2}}(5/3)$.

(C) $\dfrac{\log_2 5}{\log_2 3}$

(D) $\log_2 5$.

(15) The function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = e^{x^2/2} \int_0^x e^{-t^2/2}\, dt$$

is

(A) monotone increasing.

(B) monotone decreasing.

(C) constant.

(D) periodic.

Part B

- Part B consists of 25 questions, each carrying 5 marks.
- Answer **any 14** questions.

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(1) Let l and m be two positive integers. If the equation $1 + z^l + z^m = 0$ has a root z_0 on the unit circle, then show that z_0 is a root of unity.

(2) Let $Q(x, y)$ be a polynomial symmetric in x and y, i.e., $Q(x, y) = Q(y, x)$. If $x - y$ is a factor of $Q(x, y)$, then show that $(x - y)^2$ is also a factor of $Q(x, y)$.

(3) Let G and H be finite groups so that $(o(G), o(H)) = 1$, i.e., the order of G and the order of H are relatively prime. If K is a subgroup of the product group $G \times H$ and $(a, b) \in K$ then show that $(a, e) \in K$, where e is the identity element of H.

(4) Find all the real solutions of the system

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy.$$

(5) Express the value of

$$\binom{2004}{0}\binom{2004}{3}\binom{2004}{6}+$$

in the form $\dfrac{a^b + c}{d}$ where a, b, c and d are positive integers.

(6) Find the minimum value of

$$\frac{(x + \frac{1}{x})^6 - (x^6 + \frac{1}{x^6}) - 2}{(x + \frac{1}{x})^3 + (x^3 + \frac{1}{x^3})}$$

for $x > 0$.

(7) Find the eigenvalues of the $n \times n$ real matrix

$$\begin{matrix} 0 & b & b & & b \\ b & 0 & b & & b \\ & & & & \\ b & b & b & & 0 \end{matrix}$$

(8) Consider the system of equations in x_1, x_2, x_3 and x_4

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 \end{aligned}$$

If (c_1, c_2, c_3, c_4) and (d_1, d_2, d_3, d_4) are two distinct solutions of the system, then show that the system has infinitely many solutions.

(9) Let $(\mathbb{C} \setminus \{0\}, \cdot)$ be the multiplicative group of all non-zero complex numbers. If G is a finite subgroup of $(\mathbb{C} \setminus \{0\}, \cdot)$, then show that G is cyclic.

(10) If $x + \frac{1}{x} = u$ and $x^3 = v$, then find polynomials $P(u, v)$ and $Q(u, v)$ such that $x^2 = P(u, v)/Q(u, v)$.

(11) (A) If P, Q, R and S are real polynomials, then find real polynomials T and U so that

$$((P(x))^2 + (Q(x))^2)((R(x))^2 + (S(x))^2) = (T(x))^2 + (U(x))^2$$

3 Marks

(B) Suppose $f(x) = ax^4 + bx^3 + cx^2 + dx + e > 0$ for all $x \in \mathbb{R}$ where a, b, c, d, e are real constants. Show that $f(x) = (A(x))^2 + (B(x))^2$ for some **real** polynomials A and B.

2 Marks

(12) (A) Find the area A of the region in the first quadrant bounded by the ellipse $x^2 + 9y^2 = 9$, the line $y = mx$ and the y-axis. 3 Marks

(B) Let B be the area in the first quadrant of the region bounded by the same ellipse, the line $y = 2x$ and the x-axis. If $B = A$, then find m. 2 Marks

(13) Let (α, β) be a point on the hyperbola $x^2 - y^2 = a^2$ with $a > 0$. Let L be the tangent line to the hyperbola at (α, β). Let Q be the foot of the perpendicular dropped on the line L from the origin.

(A) Find the equation to the locus of Q. $(x^2+y^2)^2 = a^2(x^2-y^2)$ / $r^2 = a^2\cos 2\theta$ 3 Marks

(B) Draw the locus of Q. 2 Marks

(14) (A) Show that $a(x-y)(y-z)+b(y-z)(z-x)+c(x-y)(z-x) = 0$, where $(a, b, c) \neq (0, 0, 0)$, represents a pair of planes. 3 Marks

(B) When the planes are distinct, find the line of intersection. 2 Marks

(15) Let $\vec{a} \neq \vec{0}$ and $\vec{b}$ be two perpendicular vectors in $\mathbb{R}^3$ and let k be a real constant.

(A) Find a vector $\vec{x}$ such that $\vec{a} \cdot \vec{x} = k$ and $\vec{a} \times \vec{x} = \vec{b}$. 4 Marks

(B) Is the vector $\vec{x}$ unique? 1 Mark

(16) Verify Green's theorem for the line integral $\int_C x^2 dx + xy dy$, where C is the boundary of the region bounded by the x-axis, the line $x = y$ and the line $x + y = 2$.

(17) (A) Prove that $\sinh x > x$ for all $x > 0$. 2 Marks

(B) Prove that for $a, b > 0$ and $a \neq b$,

$$\frac{a - b}{\log_e a - \log_e b} > \sqrt{ab}.$$

3 Marks

(18) Let X be a finite set and let $f : X \to X$ be a function. Let f^n denote $f \circ f \circ \cdots \circ f$ (n times).

(A) Let $a \in X$. If there exists an integer $n > 1$ such that $f^n(x) = a$ for every $x \in X$, then show that $f(a) = a$. 2 Marks

(B) If for each $x \in X$, there is an n (depending on x) such that $f^n(x) = x$, the

that f is a bijection.

(19) Evaluate

$$\int_2^4 \frac{\sqrt{\log_e(9-x)}}{\sqrt{\log_e(9-x)} + \sqrt{\log_e(x+3)}} dx.$$

(20) (A) Find a function f such that

$$\tan^{-1}\left(\frac{1}{x^2+x+1}\right) = f(x) - f(x+1)$$

for all real $x \geq 1$.

(B) Hence or otherwise find the sum

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

(21) Suppose a particle is moving along the graph of $y = \log_e x$. Find a point on its tr which is closest to the point $(0, 1)$ and show that it is unique.

(22) Prove that the image of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under the mapping $f(z) = z^2$ $z = x + iy$, is also a conic. Find its centre, eccentricity and foci.

(23) Let $f : (-1, 2) \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and f' is a increasing function in the interval $[0, 1]$. Show that the function

$$g(x) = \frac{f(x)}{x} : (0, 1] \to \mathbb{R}$$

is also a strictly increasing function.

(24) (A) Let $\{x_n\}_{n\geq1}$ and $\{y_n\}_{n\geq1}$ be two real sequences having a common limit l. Pr the sequence

$$\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots\}$$

has the same limit l.

(B) Hence or otherwise prove that a function $f : \mathbb{R} \to \mathbb{R}$ which maps convergent s to convergent sequences, is continuous on $\mathbb{R}$.

(25) (A) Given that $y = e^x$ is a solution of the homogeneous equation

$$xy'' - (1+x)y' + y = 0,$$

find another linearly independent solution.

(B) Hence solve the inhomogeneous equation

$$xy'' - (1+x)y' + y = x^2 e^x.$$