Integrated Ph.D./Mathematical Sciences

General Instructions

(1) The question paper has 50 multiple choice questions.

(2) Four possible answers are provided for each question and only one of these is correct.

(3) Each question carries 2 marks.

(4) There is no negative marking.

(5) Answers are to be marked in the OMR sheet provided.

(6) For each question darken the appropriate bubble to indicate your answer.

(7) Use only HB pencils for bubbling purpose.

(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

Notations : The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

1. Let a, b be two real numbers such that $a > 0$ and $b > 0$. The number of real roots of the cubic $ax^3 + bx + 1 = 0$ is

   (A) 0

   (B) 1

   (C) 2

   (D) 3 .

2. Let α, β, γ be the roots of the cubic $x^3 + ax^2 + bx + c = 0$ where a, b, c are real. The expression $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ is equal to

   (A) $b^2 - 2ac$

   (B) $b^2 - 4ac$

   (C) $b^2 + 2ac$

   (D) $b^2 + 4ac$.

3. The equation $x^{10} + 5x^3 + x - 15 = 0$ has

   (A) at least 2 positive real roots

   (B) at least 2 negative real roots

   (C) all real roots

   (D) at least 8 imaginary roots .

4. For real numbers $x > 1$ and $y > 1$, define P, Q as

   $$P = \ln \sqrt{xy}, \quad Q = \sqrt{\ln x \ln y} \, .$$

   Which of the following is true for all $x > 1$ and $y > 1$?

   (A) $P \geq Q$

   (B) $P \leq Q$

   (C) $P = Q$

   (D) There is no relation between P, Q.

5. If $x \neq 0, y \neq 0$, then $x^2 + xy + y^2$ is

   (A) always negative

   (B) always positive

   (C) zero

   (D) sometimes positive, sometimes negative.

6. The sum $\sum_{k=0}^{n} \binom{n}{k} \binom{k}{n} x^k (1-x)^{n-k}$ is equal to

(A) x^n
(B) 1
(C) x^{2n}
(D) 0.

7. Let $z = x + iy$ be a complex number. Then $|z| = |x| + |y|$ holds if and only if

(A) $z = 0$
(B) z lies on the x-axis
(C) z lies on the y-axis
(D) z lies either on the x-axis or on the y-axis.

8. One of the values of $\arg(\sqrt{3} - \imath)^6$ is

(A) π
(B) $\pi/3$
(C) $2\pi/3$
(D) $5\pi/3$.

9. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \cos z$. Then

(A) $|f(z)| \leq 1$
(B) $|f(z)| \leq \pi$
(C) $|f(z)| \leq |z|$
(D) f is unbounded.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \bar{z}$. Then

(A) f is differentiable everywhere
(B) f is nowhere differentiable
(C) f is differentiable everywhere except at the origin
(D) f is an entire function.

11. The determinant $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$ evaluates to

(A) xy
(B) $(xy)^2$
(C) $(1-x^2)(1-y^2)$
(D) $x^2 + y^2$.

12. The determinant $\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_0 & x & x_2 & x_3 & x_4 \\ x_0 & x_1 & x & x_3 & x_4 \\ x_0 & x_1 & x_2 & x & x_4 \\ x_0 & x_1 & x_2 & x_3 & x \end{vmatrix}$ evaluates to

(A) $[x_0(x - x_1)(x - x_2)(x - x_3)(x - x_4)]^4$
(B) $x_0(x - x_1)(x - x_2)(x - x_3)(x - x_4)$
(C) $x_0[(x - x_1)(x - x_2)(x - x_3)(x - x_4)]^4$
(D) $xx_0x_1x_2x_3x_4$.

13. The number of reflexive relations on a set of cardinality 3 is

(A) 64
(B) 32
(C) 8
(D) 4.

14. Up to isomorphism, the number of groups of cardinality 4 is

(A) one and it is abelian
(B) two – one is abelian and the other non-abelian
(C) two – both are abelian
(D) four – two abelian and two non-abelian.

15. Suppose G is a group with more than one element and no proper subgroup. Then the cardinality of G is

P a prime number.
Q a finite non prime number.
R infinite.

(A) P only
(B) P or Q, but not R
(C) P or R, but not Q
(D) any of P, Q or R.

16. The number of roots of the polynomial $x^3 - x$ in $\mathbb{Z}/6\mathbb{Z}$ is

(A) 1
(B) 2
(C) 3
(D) 6.

17. Let S and T be vector subspaces of a vector space V. Then $S \cup T$ is a subspace of V

(A) is never true

(B) if and only if one of S or T is trivial

(C) if and only if $S \subseteq T$ or $T \subseteq S$

(D) if and only if $S \cap T$ is a nonzero vector space.

18. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y, z) = (y + z, z + x, x + y)$. The matrix of T with respect to the basis $\{(1, 1, 1), (1, -1, 0), (1, 1, -2)\}$ is

(A) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

(C) $\begin{pmatrix} 2 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix}$

(D) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

19. Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then T is a group homomorphism

(A) only if $\dim V \leq \dim W$

(B) only if $\dim V \geq \dim W$

(C) only if $\dim V = \dim W$

(D) is always true.

20. Minimum of dimension of the intersection of two seven dimensional vector subspaces in a twelve dimensional vector space is

(A) 0

(B) 2

(C) 5

(D) 7.

21. Dimension of kernel (i.e., null space) of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by the matrix $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ with respect to the standard bases is

(A) 0
(B) 1
(C) 2
(D) 3.

22. Let P be an $n \times n$ matrix with real entries such that

$$P^2 + 2P + I = 0$$

where I denotes the $n \times n$ identity matrix. Which of the following is true?

(A) There does not exist a matrix P satisfying the given condition
(B) $P = -I$
(C) P exists and is invertible
(D) P exists but it may not always be invertible.

23. Let Δ be the triangle in $\mathbb{R}^2$ with vertices at $(0,0), (0,1), (1,0)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by $T(x, y) = (2x + 3y, -x + 4y)$. The ratio

$$\frac{\text{area } T(\Delta)}{\text{area } \Delta}$$

is equal to

(A) 11
(B) 12
(C) 13
(D) 14

24. Let $A = (3, -1, 2)$ and $B = (0, 2, -1)$. Then the locus of points $P = (x, y, z)$ that satisfy

$$\text{distance}(PA) = 2 \text{ distance}(PB)$$

is given by

(A) $(x + 1)^2 + (y - 3)^2 + (z + 2)^2 = 12$
(B) $(x - 1)^2 + (y + 3)^2 + (z - 2)^2 = 12$
(C) $(x + 1)^2 + (y - 3)^2 + (z - 2)^2 = 12$
(D) $(x - 1)^2 + (y - 3)^2 + (z + 2)^2 = 12.$

25. Let T be the graph of the function

$$f(x) = \begin{cases} 1+x, & -1 \leq x \leq 0; \\ 1-x, & 0 \leq x \leq 1. \end{cases}$$

Then the reflection of T in the line $y = 0$ is given by the graph of $g(x)$ where

(A) $g(x) = \begin{cases} -1-x, & -1 \leq x \leq 0 \\ -1+x, & 0 \leq x \leq 1 \end{cases}$

(B) $g(x) = \begin{cases} -1+x, & -1 \leq x \leq 0 \\ -1-x, & 0 \leq x \leq 1 \end{cases}$

(C) $g(x) = \begin{cases} -1-x, & -1 \leq x \leq 0 \\ 1-x, & 0 \leq x \leq 1 \end{cases}$

(D) $g(x) = \begin{cases} 1-x, & -1 \leq x \leq 0 \\ -1-x, & 0 \leq x \leq 1. \end{cases}$

26. In the Euclidean space $\mathbb{R}^3$, the nonempty intersection of a plane with the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is

P a circle.
Q an ellipse.
R a single straight line.
S a pair of parallel staright lines.

(A) P, Q, R but not S
(B) P, Q, S but not R
(C) P, R, S but not Q
(D) Any of P, Q, R or S.

27. Suppose there are two unit circles in the Euclidean plane such that center of one is a point of the circumference of the other. Distance between the points of intersection of the circles is

(A) 2 units
(B) $\sqrt{2}$ units
(C) $1/\sqrt{3}$ units
(D) $\sqrt{3}$ units.

28. The number of points in the Euclidean plane together with the three points $(1, -1), (-5, 9)$ and $(7, -11)$ which form a parallelogram is

(A) 0
(B) 1
(C) 2
(D) infinite.

29. The equation of the tangent plane to the surface $x^2 - y^2 + xz = 2$ at the point $(1, 0, 1)$ is given by

(A) $3x - 2 - z = 0$

(B) $3x + 3 + z = 0$

(C) $3x - 4 + z = 0$

(D) $3x - 5 - z = 0$.

30. Let $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = ax^2y + by^3$, where a, b are real constants. The family of curves given by $\{u(x, y) = \text{constant}\}$ and $\{v(x, y) = \text{constant}\}$ are orthogonal exactly when

(A) $a + 3b = 0$

(B) $a - 3b = 0$

(C) $3a + b = 0$

(D) $3a - b = 0$.

31. Let $\vec{X}, \vec{Y}, \vec{Z}$ be vectors in $\mathbb{R}^3$ such that

$$\vec{X} \times \vec{Y} = \vec{\imath} + 2\vec{\jmath} - 3\vec{k}, \quad \vec{Z} = -\vec{\imath} - 2\vec{\jmath} + \vec{k}.$$

The volume of the parallelepiped in $\mathbb{R}^3$ spanned by $\vec{X}, \vec{Y}, \vec{Z}$ is

(A) 5

(B) 6

(C) 7

(D) 8.

32. Let $\vec{v} = (2xyz)\,\vec{\imath} + (x^2z + y)\,\vec{\jmath} + (x^2y + 3z^2)\,\vec{k}$. Then the magnitude of curl $\vec{v}$ at $(1, 1, 1)$ is

(A) not defined

(B) strictly greater than one

(C) equal to one

(D) equal to zero.

33. Let D be the square in $\mathbb{R}^2$ with vertices at $(0, 0), (1, 0), (0, 1), (1, 1)$. The integral

$$\int_{\partial D} x \, dy$$

where ∂D is the boundary of the square, is equal to

(A) 0

(B) 0.5

(C) 1

(D) 1.5.

34. The integral $\int_C (yz\,dx + (xz+1)\,dy + xy\,dz)$, where C is a simple closed curve, equals

(A) 0
(B) $3xyz + y$
(C) length of C
(D) area enclosed by C.

35. The value of

$$lim_{n\to\infty}\Big(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\Big)$$

is

(A) 0
(B) $\ln 2$
(C) e
(D) e^2.

36. Let $S_k = \sum_{n=2}^{k} \frac{(-1)^n}{n \ \ln \ n}$. Then the sequence $\{S_k\}$

(A) converges to a finite number
(B) diverges to ∞
(C) diverges to $-\infty$
(D) oscillates.

37. The equation $x^2 = x\sin x + \cos x$ is true for

(A) no real value of x
(B) exactly one real value of x
(C) exactly two real values of x
(D) infinitely many real values of x.

38. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x) - f(y)| \leq K|x-y|^{\frac{5}{3}}$, for all x, y, where K is a constant. Then

(A) f is a linear function
(B) f is a constant
(C) f is strictly increasing
(D) f is strictly decreasing.

39. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \min(|x|, x^2 - 1)$. Then f is

(A) a discontinuous function

(B) continuous and differentiable everywhere

(C) differentiable everywhere except at one point

(D) differentiable everywhere except at two points.

40. At $x = 2, f(x) = x^2 e^{-x}$ has a

(A) local minimum, but not global minimum

(B) local maximum, but not global maximum

(C) global minimum

(D) global maximum.

41. Let $f : [a, b] \to \mathbb{R}$ be continuous, $f(a) \geq b,\ f(b) \leq a$. Then there exists an $x \in [a, b]$ such that

(A) $f(x) = x$

(B) $f(x) = 0$

(C) $f'(x) = 0$

(D) $f''(x) = 0$.

42. Let $g(x) = \int_{x-\alpha}^{x+\alpha} \sin y^2 \, dy$. Then $g'(x)$ equals

(A) $\sin x^2$

(B) $\frac{\sin(x+\alpha)^2 + \sin(x-\alpha)^2}{2}$

(C) $\sin(x + \alpha)^2 - \sin(x - \alpha)^2$

(D) $\cos(x + \alpha)^2 - \cos(x - \alpha)^2$.

43. The double integral $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} \, dx \, dy$ equals

(A) $e + 1$

(B) 1

(C) $e - 1$

(D) e^2.

44. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x) = \int_0^x f(y)\,dy$, then

(A) $f(x) = e^x$

(B) $f(x) = \ln x$

(C) f is identically zero

(D) f is identically equal to 1.

45. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to 0} f(x) = a$. Then $\lim_{x\to 0} \frac{1}{x}\int_0^x f(y)\,dy$

(A) equals 1

(B) equals a

(C) equals -1

(D) does not exist.

46. The partial derivative of $\int_0^{x+y} \sin^2(t+y)\,dt$ with respect to x is

(A) $\sin^2(x+2y)$

(B) $2\sin(x+y)$

(C) $2\sin(x+2y)$

(D) $2\cos(x+2y)$.

47. The initial value problem $\frac{dy}{dx} = 2y^{\frac{1}{3}}$, $y(0) = 0$, has

(A) no solution

(B) infinitely many solutions

(C) exactly one solution

(D) finitely many solutions.

48. Which of the following pair of functions is not a linearly independent pair of solutions of $y'' + 9y = 0$?

(A) $\sin 3x,\ \sin 3x - \cos 3x$

(B) $\sin 3x + \cos 3x,\ 3\sin x - 4\sin^3 x$

(C) $\sin 3x,\ \sin 3x \cos 3x$

(D) $\sin 3x + \cos 3x, 4\cos^3 x - 3\cos x$.

49. Determine the type of the following differential equation $\frac{d^2y}{d^2x} + \cos(x+y) = \sin x$.

(A) linear, homogeneous

(B) nonlinear nonhomogeneous

(C) linear, nonhomogeneous

(D) nonlinear, nonhomogeneous.

50. The solution of the first order ODE

$$xy' = xy + x + y + 1$$

is (in all the choices below, C is a constant)

(A) $y = Cx(e^x - 1)$
(B) $y = (Cxe^x) - 1$
(C) $y = (Ce^x) - x$
(D) $y = (Ce^x) - x - 1$.