INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2006

Program : Integrated Ph.D

Entrance Paper : Mathematical Sciences
Paper Code : MS

Day & Date
SUNDAY, 30th APRIL 2006

Time
2.00 P.M. TO 5.00 P.M.

Integrated Ph.D. (Mathematical Sciences)

General Instructions

(1) The question paper has 50 multiple choice questions.

(2) Four possible answers are provided for each question and only one of these is correct.

(3) **Marking scheme:** Each correct answer will be awarded 2 marks, but **0.5 marks will be deducted** for each **incorrect** answer.

(4) Answers are to be marked in the OMR sheet provided.

(5) For each question darken the appropriate bubble to indicate your answer.

(6) Use only HB pencils for filling in the bubbles.

(7) Mark only one bubble per question. If you mark more than one bubble, your response will be evaluated as incorrect.

(8) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

Notations: The set of natural numbers, integers, rational numbers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

1. The number of positive roots of $x^6 + 9x^5 + 2x^3 - x^2 - 2$ is

   (A) 0.
   (B) 1.
   (C) 3.
   (D) 5.

2. Let $f(x)$ be a polynomial with real coefficients and let $f'(x)$ denote its derivative. Then, between two consecutive roots of $f'(x) = 0$, there

   (A) never is a root of $f(x) = 0$.
   (B) always is a root of $f(x) = 0$.
   (C) is at most one root of $f(x) = 0$.
   (D) may be any number of roots of $f(x) = 0$.

3. Let $\alpha, \beta, \gamma, \delta$ be roots of the quartic $x^4 + px^3 + qx^2 + rx + s$ where p, q, r and s are real. Then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ equals

   (A) $p^2 + 4q$.
   (B) $p^2 - 4q$.
   (C) $p^2 + 2q$.
   (D) $p^2 - 2q$.

4. Let $z = x + \imath y$ be a complex number. Then $|\, z \,| \; = \; |\, x \,| - \; |\, y \,|$ holds if and only if

   (A) $z = 0$.
   (B) z is real.
   (C) z is purely imaginary.
   (D) z is real or purely imaginary.

5. Let $P = re^{i\theta}, Q = r$ and $R = P + Q$. If O is the origin, then OPQR is a square if and only if

(A) $r = 0$.

(B) $r = 1$.

(C) $\theta = \pm\frac{\pi}{2}$.

(D) $\theta = \pm\pi$.

6. The determinant $\begin{vmatrix} a+b & c+d & e & 1 \\ b+c & d+a & f & 1 \\ c+d & a+b & g & 1 \\ d+a & b+c & h & 1 \end{vmatrix}$ evaluates to

(A) 0.

(B) 1.

(C) $(a+b)(c+d)+e+f+g+h$.

(D) $(a+b+c+d)(e+f+g+h)$.

7. Let $M_2(\mathbb{C})$ be the vector space of 2×2 matrices over $\mathbb{C}$.

P: The determinant function from $M_2(\mathbb{C})$ to $\mathbb{C}$ is a linear transformation.

Q: The determinant function is a linear function of each row of the matrix when the other row is held fixed.

(A) Both P and Q are true.

(B) P is true, but Q is false.

(C) P is false, but Q is true.

(D) Both P and Q are false.

8. Let x, y and z be positive real numbers such that $xyz = 1$. Then

(A) $x+y+z \geq 3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 3$.

(B) $x+y+z \geq 3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq 3$.

(C) $x+y+z \leq 3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 3$.

(D) $x+y+z \leq 3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq 3$.

9. Let F_n be a finite set with n elements. The number of one-to-one maps from F_5 to F_7 is

(A) 35 .
(B) $\binom{7}{5}$.
(C) 5!
(D) $5!\binom{7}{5}$.

10. Consider the functions $f, g : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n + 2$ and $g(n) = n^2 - 5$.

(A) Neither f nor g is a one-to-one function.
(B) The function f is one-to-one, but not g.
(C) The function g is one-to-one, but not f.
(D) Both f and g are one-to-one functions.

11. The set of integers under subtraction is not a group because

(A) subtraction is not associative.
(B) there is no identity element for subtraction.
(C) every element does not have an inverse.
(D) subtraction is not commutative.

12. Consider $\mathbb{R} \setminus \{0\}$ as a multiplicative subgroup of $\mathbb{C} \setminus \{0\}$. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $f(z) = z^2$ and g the restriction of f to $\mathbb{R} \setminus \{0\}$. Then

(A) neither f nor g is a surjective (i.e. onto) homomorphism.
(B) f is a surjective homomorphism but g is not.
(C) g is a surjective homomorphism but f is not.
(D) both f and g are surjective homomorphisms.

13. The number of distinct homomorphisms from $\mathbb{Z}/5\mathbb{Z}$ to $\mathbb{Z}/7\mathbb{Z}$ is

(A) 0.
(B) 1.
(C) 7.
(D) 5.

14. Let S_3 denote the group of permutations on the set $\{1, 2, 3\}$ and $G = S_3 \times S_3$. The set $H := \{(\sigma, \tau) \mid \sigma(1) = \tau(1)\}$ is

(A) not a subgroup of G.

(B) a non-abelian subgroup of G

(C) an abelian subgroup of G.

(D) a normal subgroup of G.

15. The set $\{0, 2, 4\}$ under addition and multiplication modulo 6 is

(A) not a ring with unity (identity).

(B) a ring with 0 as unity (identity).

(C) a ring with 2 as unity (identity).

(D) a ring with 4 as unity (identity).

16. Suppose a and b are elements in R, a commutative ring with unity. Then the equation $ax = b$

(A) always has exactly one solution.

(B) has a solution only if a is a unit.

(C) has more than one solution only if $b = 0$.

(D) may have more than one solution.

17. For what values of r is the vector $(3, 2, r, 0)$ in $\mathbb{R}^4$ contained in the subspace generated by $(1, 0, 0, 0), (0, 1, 2, 0)$ and $(0, 1, 1, 1)$?

(A) For no value of r.

(B) For exactly one value of r.

(C) For more than one but finitely many values of r.

(D) For infinitely many values of r.

18. Denote by M a real 3×3 real matrix such that $M \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $M \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/5 \\ 0 \end{pmatrix}$, then $M \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ is

(A) $\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1/6 \\ 0 \\ 0 \end{pmatrix}$

(C) $\begin{pmatrix} 2 + \frac{1}{4} \\ 0 \\ 0 \end{pmatrix}$

(D) not determined uniquely.

19. Suppose u, v and w are linearly dependent vectors and w is not expressible as a linear combination of u and v. Then which of the following statements is true?

(A) Such a situation is not possible.

(B) Vector w has to be zero.

(C) One of u or v has to be zero.

(D) Vector u is a multiple of v.

20. In a 13 dimensional vector space, the dimension of intersection of two 6 dimensional subspaces is

(A) at least 1.

(B) at most 1.

(C) at least 6.

(D) at most 6.

21. Which of the options is true about the following statement?

Let $T : \mathbb{R}^5 \to \mathbb{R}^7$ be a linear transformation whose kernel is of dimension 2 and whose image is a line.

(A) There does not exist any such linear transformation.

(B) There is exactly one such linear transformation.

(C) There are finitely many (but more than one) such linear transformations.

(D) There are infinitely many such linear transformations.

22. For maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, match the following

P.	$f(x, y) = (x, x - y, y)$	1.	Not a linear transformation.
Q.	$g(x, y) = (x, x, x)$	2.	Rank two linear transformation.
R.	$h(x, y) = (1, x + y, x - y)$	3.	Rank one linear transformation.

(A) P-3, Q-2, R-1.

(B) P-3, Q-1, R-2.

(C) P-2, Q-1, R-3.

(D) P-2, Q-3, R-1.

23. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be a linear transformation. Then T has

(A) no real eigenvalues.

(B) at least one real eigenvalue.

(C) at most one real eigenvalue.

(D) exactly one real eigenvalue.

24. A non-zero linear transformation T on $\mathbb{R}$

(A) may not have any eigenvector.

(B) has exactly one eigenvector.

(C) has more than one (but finitely many eigenvectors).

(D) has infinitely many eigenvectors.

25. Let X and Y denote spheres in $\mathbb{R}^3$ each with radius 2 and center at $(0, 1, 0)$ and $(0, -1, 0)$ respectively. Then $X \cap Y =$

(A) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 1\}$.

(B) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 0\}$.

(C) $\{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = 1,\ x = 0\}$.

(D) $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 + x^2 = 1,\ y = 0\}$.

26. The set of points in the Euclidean plane satisfying the quadratic $x^2 + y^2 + x + y + 1 = 0$ is

(A) an empty set.

(B) a pair of straight lines.

(C) a circle.

(D) an ellipse.

27. Let $\boldsymbol{V}, \boldsymbol{W}$ and $\boldsymbol{X}$ be three vectors in $\mathbb{R}^3$, and let $\boldsymbol{.}$ and $\times$ denote the usual dot product and cross product respectively. The product $\boldsymbol{X} \times (\boldsymbol{Y} \boldsymbol{.} \boldsymbol{Z})$

(A) is a vector in $\mathbb{R}^3$.

(B) gives the volume of the parallelepiped spanned by $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$.

(C) always equals 0.

(D) cannot be defined.

28. The equation of the tangent plane to the surface $x^2 - 3xy + 2y^2 + z^2 = 1$ at the point $(1, 1, 1)$ is given by

(A) $x + y + 2z = 4$.

(B) $x - y - 2z = -2$.

(C) $x - 2y - 2z = -2$.

(D) $x + y + z = 3$.

29. Let $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ be vectors in $\mathbb{R}^3$ such that

$$\boldsymbol{X} \times \boldsymbol{Y} = 2i - 2j + 5\boldsymbol{k}, \qquad \boldsymbol{Y} = i + 3j.$$

The volume of the parallelepiped in $\mathbb{R}^3$ spanned by $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ is

(A) 4.
(B) 3.
(C) 2.
(D) 1.

30. Let x and y be any two real numbers satisfying $0 < x < 1 < y$. Then, the limit

$$\lim_{n \to \infty} \left(x + \frac{y}{n} \right)^n$$

is equal to

(A) $e^{y/x}$.
(B) $e^{x/y}$.
(C) 0.
(D) e.

31. The limit $\lim_{n \to \infty} (n^5 + 4n^3)^{1/5} - n$ equals

(A) 4.
(B) 4/5.
(C) 0.
(D) 5/4.

32. We are given a convergent series $\sum_{n=1}^{\infty} a_n$, where $a_n \geq 0$ for each n. Which of the following correctly describes the behaviour of the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^p}, \quad 1 \leq p \leq 2 \ ?$$

(A) Diverges when $p = 1$, but converges for $p \in (1, 2]$.
(B) Converges for every $p \in [1, 2]$.
(C) Diverges when $p \in [1, 5/4]$, but converges for $p \in (5/4, 2]$.
(D) Diverges for every $\in [1, 2]$.

33. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} ax^2 + b, & \text{if } x \le 1, \\ cx + 1, & \text{if } x > 1. \end{cases}$$

We want to find appropriate values of a, b and c such that

I) f is increasing in the interval $(0, \infty)$; and
II) f' is continuous on $\mathbb{R}$.

Which of the following is the correct statement about the values of (a, b, c) for which both conditions (I) and (II) are satisfied ?

(A) $(3, 2, 6)$ is the only possible value.
(B) There are finitely many values of (a, b, c).
(C) $(-2, -3, -4)$ is one of the values.
(D) There are infinitely many values of (a, b, c).

34. The limit

$$\lim_{x \to 0} \frac{3^x - 2^x}{4^x - 3^x}$$

equals

(A) $3/4$.
(B) $\log_{4/3}(3/2)$.
(C) $\log_e(3/2)$.
(D) $2/3$.

35. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \min(3x^3 + x, |x|)$ is

(A) continuous on $\mathbb{R}$, but not differentiable at $x = 0$.
(B) differentiable on $\mathbb{R}$, but f' is discontinuous at $x = 0$.
(C) differentiable on $\mathbb{R}$, and f' is continuous on $\mathbb{R}$.
(D) differentiable to any order on $\mathbb{R}$.

36. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and define

$$g(x) = \int_0^{x^3+3x^2} f(t)dt.$$

The value of $g'(0)$

(A) equals 0.

(B) equals 1.

(C) is a positive real number.

(D) cannot be determined without knowing the value of $f(0)$.

37. Let $f(x)$ be a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$. Suppose that the graph of f intersects the x-axis at exactly two distinct points and that $\lim_{x\to\infty} f(x) = -\infty$. Then:

(A) f has a unique point of global minimum in $\mathbb{R}$.

(B) f has a unique point of global maximum in $\mathbb{R}$.

(C) f has exactly two points of local maximum in $\mathbb{R}$.

(D) f has exactly one point of local maximum in $\mathbb{R}$.

38. For any $x \in \mathbb{R}$, let $[x]$ denote the greatest integer smaller than or equal to x. The value of the integral

$$\int_{1/2}^{1} [1/x^2]dx$$

equals

(A) $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} - \frac{1}{2}$.

(B) $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}$.

(C) $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} + \frac{1}{2}$.

(D) $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}$.

39. The value of the integral

$$\int_0^1 \int_{\sqrt{y}}^1 3\sqrt{x^3+1}\, dxdy$$

is

(A) $2^{5/2}/3$.
(B) $\frac{2}{3}(2^{3/2}-1)$.
(C) $\frac{2}{3}(\sqrt{2}-1)$.
(D) $2(\sqrt{2}-1)$.

40. Define

$$F(x,y) = \int_0^{x^2+y^2} \cos^2(t+x)dt, \quad (x,y) \in \mathbb{R}^2.$$

$\dfrac{\partial F}{\partial x}(0,y)$ equals

(A) 0.
(B) $\dfrac{\cos(y^2)}{2}$.
(C) $\dfrac{\cos(y^2)}{2} + \cos^2(y^2)$.
(D) $\cos^2(y^2) + \cos(y^2)$.

41. The vector field $\boldsymbol{V}(x,y) = ye^{2x}\boldsymbol{i} + e^{2x}\boldsymbol{j}$

(A) is a conservative vector field on $\mathbb{R}^2$ having the potential function $\phi(x,y) = ye^{2x}/2$.
(B) is a conservative vector field on $\mathbb{R}^2$.
(C) has the property that the work done by $\boldsymbol{V}$ along every path is 0.
(D) is not a conservative vector field.

42. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable at each point in $\mathbb{R}^3$. Then $\mathbf{curl}[\mathbf{grad}(f)]$

(A) is a vector field that is orthogonal to the level surfaces of f.
(B) is a conservative vector field.
(C) equals $\mathbf{0}$ (i.e. the zero vector) at each point in $\mathbb{R}^3$.
(D) is a non-constant vector field on $\mathbb{R}^3$.

43. Let $\triangle$ be the triangle in $\mathbb{R}^2$ with vertices $(0,0), (1,0)$ and $(0,1)$, and let $\boldsymbol{F}(x,y) = -xy\boldsymbol{i} + \sqrt{y^4+1}\boldsymbol{j}$. The line integral of the vector field $\boldsymbol{F}$:

$$\oint_{\partial\triangle} \boldsymbol{F} \cdot dr,$$

taking the anti-clockwise orientation on $\partial\triangle$ (here, $\partial\triangle$ denotes the boundary of $\triangle$), is

(A) $-1/6$.
(B) 0.
(C) $1/6$.
(D) 6.

44. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(x,y,z) = x^2 + 2xy + 5y^2 - z^4 - 1$. The unit vector $\boldsymbol{u}$ which gives the maximum value for the directional derivative $D_{\boldsymbol{u}}f$ at the point $(1,0,1)$ is

(A) $\boldsymbol{u} = (1,0,0)$.
(B) $\boldsymbol{u} = (0,0,1)$.
(C) $\boldsymbol{u} = -\frac{1}{\sqrt{2}}(1,0,1)$.
(D) $\boldsymbol{u} = \frac{1}{\sqrt{6}}(1,1,-2)$.

45. Let G be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$. The outward flux of the vector field $\boldsymbol{V}(x,y,z) = 2z\cos(xy)\boldsymbol{i} - (z^2+1)\boldsymbol{j} + yz^2\sin(xy)\boldsymbol{k}$ across the boundary of G is

(A) $-1/6$.
(B) 0.
(C) $1/6$.
(D) 6.

46. The general solution of the first-order ODE

$$xy' + x^2y - y = 0$$

is (in all choices below, C denotes a constant)

(A) $y(x) = xe^{-x^2/2} + C$.
(B) $y(x) = e^{x^2/2}(x + C)$.
(C) $y(x) = e^{x^2/2} - Cx$.
(D) $y(x) = Cxe^{-x^2/2}$.

47. The general solution of the first-order ODE

$$xy' + 2x^2y - xe^{-x^2} = 0$$

is (in all choices below C denotes a constant)

(A) $y(x) = e^{x^2}(x + C)$.
(B) $y(x) = e^{-x^2}(x + C)$.
(C) $y(x) = xe^{x^2} + C$.
(D) $y(x) = x + C$.

48. Consider the functions $f(x) = x|x|$ and $g(x) = x^2$. Then

(A) $\{f, g\}$ is a linearly independent pair of functions on $(-\infty, 0)$.
(B) $\{f, g\}$ is a linearly independent pair of functions on $(0, \infty)$.
(C) $\{f, g\}$ is a linearly dependent pair of functions on $\mathbb{R}$.
(D) $\{f, g\}$ is a linearly independent pair of functions on $\mathbb{R}$.

49. Which of the following pair of functions is *NOT* a linearly independent pair of solutions for the second-order ODE $y'' + 9y = 0$?

(A) $(\sin 3x, \cos 3x)$
(B) $(3\cos 3x - 2\sin 3x, \sin 3x)$
(C) $(\cos^3 x - 3\sin^2 x \cos x, \cos 3x)$
(D) $(\cos 3x - 3\sin 3x, \sin 3x - 3\cos 3x)$

50. Consider the second-order ODE

$$x^2y'' + Axy' + y = 0.$$

This equation

(A) admits, for each $A > 0$, a linearly independent pair of solutions consisting of trigonometric functions.

(B) admits, for some values of $A > 0$, a linearly independent pair of solutions consisting of powers of x.

(C) does not admit any linearly independent pair of solutions consisting of powers of x for any $A > 0$.

(D) has no solutions.

⋆⋆⋆ **End of question paper** ⋆⋆⋆