Instructions

1. This question paper has fifty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.

Integrated Ph. D./ Mathematical Sciences

1. Let $f : [0,1] \to [0,1]$ be continuous and $f(0) = 0$, $f(1) = 1$. Then, f is necessarily

   (A) injective, but not surjective.
   (B) surjective, but not injective.
   (C) bijective.
   (D) surjective.

2. Let $f : (-1,1) \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Then,

   (A) f need not be differentiable at the origin.
   (B) $f'(0) > 0$.
   (C) $f'(0) = 0$.
   (D) $f'(0) < 0$.

3. Let $\alpha_1, \cdots, \alpha_{2007}$ be the roots of the equation $1 + x^{2007} = 0$. Then, the value of the product $(1 + \alpha_1) \cdots (1 + \alpha_{2007})$ is

   (A) 0.
   (B) 1.
   (C) 2.
   (D) 2007.

4. Let $q > 1$ be a positive integer. Then, the set $\{(\cos \frac{\pi}{q} + i \sin \frac{\pi}{q})^n : n = 0, 1, 2, \cdots\}$, where $i = \sqrt{-1}$, is

   (A) a singleton.
   (B) a finite set, but not a singleton.
   (C) a countably infinite set.
   (D) dense on the unit circle.

5. Consider the second order ordinary differential equation $y'' + by' + cy = 0$, where b, c are real constants. You are given that $y = \exp(2x)$ is a solution. Then,

   (A) $b^2 + 4c < 0$.

   (B) $b^2 + 4c \geq 0$.

   (C) $b^2 - 4c < 0$.

   (D) $b^2 - 4c \geq 0$.

6. Consider the second order ordinary differential equation $y'' + 3y' + 2y = 0$. Then, $\lim_{n\to\infty} y(t)$ is

   (A) a non-zero finite number.

   (B) 0.

   (C) $-\infty$.

   (D) ∞.

7. Consider the system $x' = -y$, $y' = x$ with $x(0) = 1$, $y(0) = 1$. Then,

   (A) $y = \sin t + \cos t$.

   (B) $y = -\sin t + \cos t$.

   (C) $y = t \exp t + \exp t$.

   (D) y is not any of the above.

8. Consider the equation $x^{2007} - 1 + x^{-2007} = 0$. Let m be the number of distinct complex, non-real roots and n be the number of distinct real roots of the above equation. Then, $m - n$ is

   (A) 0.

   (B) 2006.

   (C) 2007.

   (D) 4014.

9. Let a, b, c be non-zero real numbers. Then, the minimum value of

$$a^2 + b^2 + c^2 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

is

(A) 0.
(B) 6.
(C) 3^2.
(D) 6^2.

10. Consider the set $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 + 2x + 4y + 6 = 0\}$. Then, A is

(A) an infinite set.
(B) a finite set with more than one element.
(C) a singleton.
(D) an empty set.

11. Consider the sequence $\{l_n\}_{n \in \mathbb{N}}$ with $l_n = \frac{1}{n+1} + \cdots + \frac{1}{2n}$. This sequence

(A) is increasing and bounded.
(B) increases to ∞.
(C) decreases to 0.
(D) decreases to a positive number.

12. Let p be a polynomial of degree $2n + 1$ with real coefficients. We say that a real number a is a fixed point of p if $p(a) = a$. Then, p has

(A) exactly $2n + 1$ fixed points.
(B) at least one fixed point.
(C) at most one fixed point.
(D) n fixed points.

13. Let $f(x) = e^{(e^{-x})}$ and define $g(x) = f(x+1) - f(x)$. Then, as $x \to \infty$, the function $g(x)$ converges to

(A) 0.

(B) 1.

(C) e.

(D) e^e.

14. Let A, B be 2×2 matrices with real entries, and assume that $AB - BA = cI$ for some constant c, where I is the identity matrix. Then, c is

(A) 0.

(B) 1.

(C) 2.

(D) 4.

15. Let f and g be any two non-constant Riemann-integrable functions on an interval $[a, b]$. Then, $\int_a^b f(x)g(x)dx$

(A) is $(\int_a^b f(x)dx)(\int_a^b g(x)dx)$.

(B) is $f(a)(\int_a^b g(x)dx)$.

(C) is $f(a)(\int_a^b g(x)dx) + g(a)(\int_a^b f(x)dx)$.

(D) does not have a representation as above.

16. Let $A = \begin{bmatrix} a & \pi \\ \pi & 1/49 \end{bmatrix}$, where a is a real number. Then, A is invertible

(A) for all $a \neq 22^2$.

(B) for all $a \neq 180^2 \times 49$.

(C) for all $a \neq 22^2$ or $a \neq 180^2 \times 49$.

(D) for all rational a.

17. Let A be an $n \times n$ matrix with real entries and suppose that the system $Ax = 0$ has the unique solution $x = 0$. Then, the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Tx = Ax$ is

(A) a bijection.
(B) one-one, but not onto.
(C) onto, but not one-one.
(D) neither one-one nor onto.

18. If A is an $n \times n$ matrix with real or complex entries and $A^3 = 0$, then

(A) $(I + A)^3 = 0$.
(B) $I + A$ is invertible.
(C) $I + A$ is not invertible.
(D) necessarily $A = 0$.

19. Let A be an $n \times n$ invertible matrix with integer entries and assume that A^{-1} also has only integer entries. Then,

(A) $\det A = n$.
(B) $\det A = \pm 1$.
(C) $\det A = n^2$.
(D) $\det A$ will depend on the entries of A and A^{-1}.

20. The eigenvalues of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are

(A) $\cos\theta$ and $\sin\theta$.
(B) $\tan\theta$ and $\cot\theta$.
(C) $e^{i\theta}$ and $e^{-i\theta}$.
(D) 1 and 2.

21. Let $A(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$, where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are differentiable on $\mathbb{R}$. Then, $\frac{d}{dt} \det A(t)$ is

(A) $\det \begin{bmatrix} a'(t) & b'(t) \\ c'(t) & d'(t) \end{bmatrix}$.

(B) $\det \begin{bmatrix} a(t) & b'(t) \\ c(t) & d'(t) \end{bmatrix}$.

(C) $\det \begin{bmatrix} a'(t) & b(t) \\ c'(t) & d(t) \end{bmatrix}$.

(D) $\det \begin{bmatrix} a'(t) & b'(t) \\ c(t) & d(t) \end{bmatrix} + \det \begin{bmatrix} a(t) & b(t) \\ c'(t) & d'(t) \end{bmatrix}$.

22. For $n > 1$, let $f(n)$ be the number of $n \times n$ real matrices A such that $A^2 + I = 0$. Then,

(A) $f \equiv 0$.

(B) $f(n) = 0$ if and only if n is even.

(C) $f(n) = 0$ if and only if n is odd.

(D) $f \equiv \infty$.

23. Let the sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers converge to a non zero real number a and let $y_n = a - x_n$. Then $\max_{n\in\mathbb{N}}\{x_n, y_n\}$ converges to

(A) a always.

(B) 0 always.

(C) $\max\{a, 0\}$.

(D) $\min\{a, 0\}$.

24. Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial with real coefficients, where $c_0 > 0$ and $c_n < 0$. Then,

(A) $f(x) > 0$ for all $x > 0$.

(B) $f(x) < 0$ for all $x < 0$.

(C) $f(x) = 0$ for some $x > 0$.

(D) $f(x) = 0$ for infinitely many values of x.

25. Which of the following is an equivalence relation in $\mathbb{R}$:

(A) $x \leq y$ for all $x, y \in \mathbb{R}$.

(B) $x - y$ is an irrational number.

(C) $x - y$ is divisible by 3.

(D) $x - y$ is a perfect square.

26. Let X be anon-empty set. A relation $\sim$ on X is called *circular* if whenever $x \sim y$ and $y \sim z$, then $z \sim x$; and *triangular* if whenever $x \sim y$ and $x \sim z$, then $y \sim z$. An equivalence relation is

(A) circular and triangular.

(B) neither circular nor triangular.

(C) circular, but not triangular.

(D) triangular, but not circular.

27. Let f be a real differentiable function defined on $[a, b]$, where the derivative is an increasing function and $x_0 \in [a, b]$. Then,

(A) f is always strictly increasing.

(B) f is always strictly decreasing.

(C) $f(x) \leq f(x_0) + (x - x_0)f'(x_0)$ for all $x \in [a, b]$.

(D) $f(x) \geq f(x_0) + (x - x_0)f'(x_0)$ for all $x \in [a, b]$.

28. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function and define $g(x) = \displaystyle\int_0^{3x - \sin 2x} f(t)dt$. Then, the value of $g'(0)$ is

(A) 1.

(B) 0.

(C) 3.

(D) cannot be determined from the given data.

29. Let x, y and z be any 3 positive real numbers. Then, always:

(A) $\sqrt{xyz} \leq \frac{x+y+z}{3}$.

(B) $\sqrt{xyz} \geq \frac{x+y+z}{3}$.

(C) $\sqrt{xyz} \leq \left(\frac{x+y+z}{3}\right)^{3/2}$.

(D) $\sqrt{xyz} \geq \left(\frac{x+y+z}{3}\right)^{3/2}$.

30. Consider the two functions $f(x) = |x| \sin x$ and $g(x) = x \sin x$. Then, $\{f, g\}$ is

(A) linearly independent on $(-\infty, 0)$.

(B) linearly independent on $(0, \infty)$.

(C) linearly dependent on $\mathbb{R}$.

(D) linearly independent on $\mathbb{R}$.

31. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Assume that $T(x) = 0$ for all x such that $|x| = 1$. Then,

(A) $T \equiv 0$.

(B) T is onto.

(C) dimension of kernel of T is 1.

(D) dimension of range of T is 1.

32. Let A be a matrix of order 2 with real entries such that $AB = BA$ for all matrices B of order 2. Then,

(A) A is always the zero matrix.

(B) $A = \lambda I$ for some $\lambda \in \mathbb{R}$.

(C) A is always invertible.

(D) A is never invertible.

33. Consider the space $V = \{(x_1 + x_2 + x_3, x_1 + x_2, x_3) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$. Then, the dimension of V is

(A) 0.
(B) 1.
(C) 2.
(D) 3.

34. Let $n > 2$ and for $1 \leq j \leq n$, define a_j to be the vector in $\mathbb{R}^n$ with j^{th} entry 0 and the remaining entries 1. Then, $\{a_1, \cdots, a_n\}$

(A) is a linearly dependent set.
(B) is an orthogonal system.
(C) spans a proper subspace of $\mathbb{R}^n$.
(D) is a basis for $\mathbb{R}^n$.

35. Let V be a 25 dimensional vector space. Then, the dimension of the intersection of two 13 dimensional subspaces of V

(A) is always 1.
(B) can be any integer between (and including) 0 and 13.
(C) can be any integer between (and including) 1 and 13.
(D) is none of the above.

36. Let S_4 denote the symmetry group of 4 letters and $\mathbb{R}^\star$ be the multiplicative group of non-zero real numbers. If $f : S_4 \to \mathbb{R}^\star$ is a homomorphism, then the set $\{x \in S_4 : f(x) = 1\}$ has

(A) at least 12 elements.
(B) exactly 24 elements.
(C) at most 12 elements.
(D) exactly 4 elements.

37. For positive integers n and m, where $n, m > 1$, suppose that $n\mathbb{Z}$ and $m\mathbb{Z}$ are isomorphic as rings. Then,

(A) there is no restriction on n and m.

(B) $n = m$.

(C) $g.c.d(n, m) = 1$.

(D) necessarily $n|m$ or $m|n$, but not both.

38. Let $\mathbb{Z}_n$ denote the additive group of integers modulo n. Suppose $\mathbb{Z}_n \times \mathbb{Z}_m \simeq \mathbb{Z}_{mn}$. Then,

(A) $g.c.d(n, m) = 1$.

(B) $n = m = 1$.

(C) $n|m$.

(D) $mn = m + n$.

39. Let S_n be the symmetry group of n letters and assume that it is abelian. Then,

(A) $n = 1$ or $n = 2$.

(B) n is a prime greater than 2.

(C) n is an even number greater than 2.

(D) n is an odd number greater than 2.

40. Let a and b be two non-zero vectors in $\mathbb{R}^3$ such that $|a \times b| = |a|\,|b|$. Then,

(A) a and b are orthogonal.

(B) a and b are parallel.

(C) the angle between a and b is $\pi/4$.

(D) a conclusion is not possible with the given data.

41. Let a, b and c be three vectors in $\mathbb{R}^3$, Then, $(a \times b) \cdot ((b \times c) \times (c \times a))$ is

(A) $((a \times b) \cdot c)^2$.

(B) $(a \cdot (b \times c))^2$.

(C) $a \cdot (b \times c) + (a \times b) \cdot c$.

(D) is always 0.

42. Consider the two space curves given by the parametric equations $\gamma_1(t) := (t, t^2, t^3)$, for all $t \in \mathbb{R}$ and $\gamma_2(s) := (s-1, s^2+s+4, 7s-13)$ for all $s \in \mathbb{R}$. Then, they

(A) never intersect.

(B) intersect exactly at 1 point.

(C) intersect exactly at 2 points.

(D) intersect exactly at 3 points.

43. For the surface $x^2 + 9y^2 - z^2 = 16$, the tangent plane at $(4, 1, 3)$ is given by

(A) $8x + 18y - 3z = 41$.

(B) $4x + 9y - 3z = 16$.

(C) $x + 9y - z = 10$.

(D) $4x + y - 3z = 8$.

44. Let $\sigma : (-1, 1) \to \mathbb{R}^3$ be a differentiable curve such that $\sigma'(t) \cdot \sigma'(t) = 1$ for all $t \in (-1, 1)$. Then,

(A) $\sigma''(t)$ is perpendicular to $\sigma'(t)$ for all $t \in (-1, 1)$.

(B) $\sigma''(t)$ is parallel to $\sigma'(t)$ for all $t \in (-1, 1)$.

(C) $\sigma(t) = (t, 0, 0)$ for all $t \in (-1, 1)$.

(D) $\sigma(t) \cdot \sigma'(t) = t$ for all $t \in (-1, 1)$.

45. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be thrice differentiable and vanish on the boundary of the region $\Omega = (-1,1) \times (-1,1)$. Then,

$$\int_{-1}^{1}\int_{-1}^{1} \operatorname{div}(\operatorname{grad} f)(x,y)dxdy$$

is

(A) never 0.
(B) 1.
(C) 0.
(D) dependent on f.

46. Let X, Y, Z be three vectors in $\mathbb{R}^3$ such that $X = \hat{\mathbf{i}} + 2\hat{\mathbf{k}}$ and $Y \times Z = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are the standard unit vectors along the coordinate directions. Then, the volume of the parallelpiped spanned by X, Y, Z is

(A) 2.
(B) 4.
(C) 6.
(D) 8.

47. Let E be the ellipsoid $(x-1)^2 + y^2 + \frac{1}{9}z^2 = 1$ and S be the sphere with center $(1,0,4)$ and radius $\sqrt{7}$. Then, $E \bigcap S$ is

(A) an ellipse, but not a circle.
(B) the set $\{(x,y,z) : (x-1)^2 + y^2 = 3/4\}$.
(C) the set $\{(x,y,z) : (x-1)^2 + y^2 = 3/4, z = 3/2\}$.
(D) the empty set.

48. Let S be the plane whose normal vector make angles $\pi/3, \pi/4, \pi/3$ with x, y, z axes respectively. If the point $(1,1,1)$ is in S, then, the equation of S is

(A) $\sqrt{2}x + y + z = 2 + \sqrt{2}$.
(B) $x + \sqrt{2}y + z = 2 + \sqrt{2}$.
(C) $x - \sqrt{2}y + z = 1 - \sqrt{2}$.
(D) $\sqrt{2}x + y + \sqrt{2}z = 2\sqrt{2} + 1$.

49. Let x be a real number with $\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{2}}$. Then, the quantity $\frac{x}{\sqrt{2}} + \frac{\sqrt{2}}{x}$ lies in

(A) $[1, \sqrt{2})$.

(B) $[\sqrt{2}, \sqrt{3})$.

(C) $[\sqrt{3}, 2)$.

(D) $[2, \infty)$.

50. Let a_1, a_2, a_3, a_4 be any 4 consecutive binomial coefficients in the expansion of $(x + y)^n$. Then, $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4}$ is

(A) $\dfrac{2a_1}{a_1 + a_2}$.

(B) $\dfrac{2a_2}{a_2 + a_3}$.

(C) $\dfrac{2a_3}{a_3 + a_4}$.

(D) $\dfrac{2a_4}{a_4 + a_1}$.