Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.

Integrated Ph. D./ Mathematical Sciences

1. Let X be a set with 30 elements. Let A, B, C be subsets of X with 10 elements each such that $A \cap B \cap C$ has 4 elements. Suppose $A \cap B$ has 5 elements, $B \cap C$ has 6 elements, and $C \cap A$ has 7 elements, how many elements does $A \cup B \cup C$ have ?

   (A) 16.
   (B) 14.
   (C) 15.
   (D) 30.

2. If $\alpha_1, \alpha_2, \cdots, \alpha_6$ are roots of $x^6 + x^2 + 1 = 0$, then which of the following is the value of $(1 - 2\alpha_1)(1 - 2\alpha_2) \cdots (1 - 2\alpha_6)$?

   (A) 0.
   (B) 1.
   (C) 64.
   (D) 81.

3. If a, b are arbitrary positive real numbers, then the least possible value of $\frac{6a}{5b} + \frac{10b}{3a}$ is

   (A) 4.
   (B) $\frac{6}{5}$.
   (C) $\frac{10}{3}$.
   (D) $\frac{68}{15}$.

4. Let $p(x) = x^{10} + a_1x^9 + \cdots + a_{10}$ be a polynomial with real coefficients. Suppose $p(0) = -1$, $p(1) = 1$, $p(2) = -1$. Let R be the number of real zeros of $p(x)$. Which of the following must be true ?

(A) $R \geq 4$.
(B) $R = 3$.
(C) $R = 2$.
(D) $R = 1$.

5. Let $p(x)$ and $q(x)$ be non-zero polynomials with real coefficients such that degree($p(x)$) > degree($q(x)$). If the graphs of $y = p(x)$ and $y = q(x)$ intersect in 3 points, which of the following must be true ?

(A) degree($p(x)$) ≤ 2.
(B) degree($p(x)$) ≥ 3.
(C) degree($p(x)$) $= 2$.
(D) degree($p(x)$) $= 6$.

6. Let $A = \begin{pmatrix} 12 & 24 & 5 \\ x & 6 & 2 \\ -1 & -2 & 3 \end{pmatrix}$. The value of x for which the matrix A is not invertible is

(A) 6.
(B) 12.
(C) 3.
(D) 2.

7. Let a, b be arbitrary real numbers satisfying $a^2 + b^2 = 10$. The largest possible value of $|a + 2b|$ is

(A) 7.
(B) 5.
(C) $3\sqrt{10}$.
(D) $\sqrt{50}$.

8. Let $A = \begin{pmatrix} \pi & p \\ q & r \end{pmatrix}$ where p, q, r are rational numbers. If $\det A = 0$ and $p \neq 0$, then the value of $q^2 + r^2$

(A) is 2.
(B) is 1.
(C) is 0.
(D) cannot be determined using the given information.

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 real matrix with $\det(A) = 1$. If A has no real eigenvalues then

(A) $(a + d)^2 < 4$.
(B) $(a + d)^2 = 4$.
(C) $(a + d)^2 > 4$.
(D) $(a + d)^2 = 16$.

10. Let $P = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$. Suppose $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation satisfying $A(v) = \mathbf{0}$ for all $v \in P$ and also $A(0, 0, 1) = \mathbf{0}$ (here $\mathbf{0}$ denotes the vector $(0, 0, 0)$). Then

(A) The dimension of the null space of A is 2.
(B) A is the zero linear transformation.
(C) Image $A = \mathbb{R}^3$.
(D) The dimension of the image of A is 2.

11. Suppose $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $A^3 = I$, where I is the identity transformation. Then

(A) All eigenvalues of A have to be real.
(B) The product of the eigenvalues of A must be 1.
(C) Necessarily $A = I$.
(D) A need not be an invertible matrix.

12. Let the group $G = \mathbb{R}$ under addition and the group $H =$ the set of all positive real numbers under multiplication. Then

(A) H is a cyclic group.

(B) G is a cyclic group.

(C) G and H are isomorphic

(D) G and H are not isomorphic.

13. A *generator* for a group G is an element $g \in G$ such that every element of G is equal to some power of g. Let G be a cyclic group of order 7. Then the number of generators of G is

(A) 1.

(B) 3.

(C) 6.

(D) 7.

14. Let G be the set of 2×2 real matrices which are invertible. Consider G with the binary operation $\circ$ of matrix multiplication. Then

(A) $(G, \circ)$ is a finite group.

(B) $(G, \circ)$ is an infinite group.

(C) $(G, \circ)$ is an abelian group.

(D) $(G, \circ)$ is not a group.

15. Define a relation $\sim$ on $\mathbb{R}$ as follows: given $x, y \in \mathbb{R}$, $x \sim y$ iff $x - y$ is a rational number. Then

(A) Given x, there are only finitely many y such that $y \sim x$.

(B) Given x, the set of y such that $y \sim x$ is a bounded subset of $\mathbb{R}$.

(C) $\sim$ is not an equivalence relation.

(D) $\sim$ is an equivalence relation.

16. Let S denote the set of unit vectors in $\mathbb{R}^3$ and W a vector subspace of $\mathbb{R}^3$. Let $V = W \cap S$. Then

(A) V is always a subspace of $\mathbb{R}^3$.

(B) V is a subspace of $\mathbb{R}^3$ iff W has dimension 1.

(C) V is a subspace of $\mathbb{R}^3$ iff W has dimension 3.

(D) V is never a subspace of $\mathbb{R}^3$.

17. Define a sequence s_n by

$$s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}}$$

Then the limit of s_n as n tends to infinity

(A) is 0.

(B) is 1.

(C) is ∞.

(D) doesn't exist.

18. If $\lim_{x \to 0} \left(\frac{1 + cx}{1 - cx} \right)^{\frac{1}{x}} = 4$, then $\lim_{x \to 0} \left(\frac{1 + 2cx}{1 - 2cx} \right)^{\frac{1}{x}}$ is

(A) 2.

(B) 4.

(C) 16.

(D) 64.

19. Let the limits of the sequences a_n and b_n, respectively, be k and k^3. If the sequence $a_1, b_1, a_2, b_2, \cdots, \cdots$ has a limit, then the value of this limit

(A) is 0 or 1 or -1.

(B) is 0 or 1.

(C) is $k + k^3$.

(D) is k^4.

20. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Define $g : [a, b] \to \mathbb{R}$ by

$g(x) = \sup\{f(y) : y \in [a, x]\}$. Then $g(x)$

(A) must be differentiable.

(B) must be continuous and Riemann integrable.

(C) must be continuous, but not Riemann integrable.

(D) need not be continuous.

21. If p is a real polynomial with $p(0) = 1$ and $p'(x) > 0$ for all x then

(A) p has more than one real zero.

(B) p has exactly one positive zero.

(C) p has exactly one negative zero.

(D) p has no real zero.

22. If $y = f(x)$ satisfies the differential equation $y' = \cos y$, $y(0) = 0$ then

(A) $|f(x)| \leq x^2$.

(B) $|f(x)| \leq |x|$.

(C) $|f(x)| \leq |\sin x|$.

(D) $|f(x)| \leq |\cos x|$.

23. For a square matrix A, let $tr(A)$ denote the sum of its diagonal entries. Let I denote the identity matrix. If A and B are 2×2 matrices with real entries such that $\det(A) = \det(B) = 0$ and $tr(B) \neq 0$, then the limit of $\dfrac{\det(A + tI)}{\det(B + tI)}$ as $t \to 0$ is

(A) zero.

(B) infinity.

(C) $\dfrac{tr(A)}{tr(B)}$.

(D) $\det(A + B)$.

24. Let $p(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$ be a polynomial. Then $\lim_{n\to\infty} n \int_0^1 x^n p(x)dx$ equals

(A) $p(1)$.

(B) $p(0)$.

(C) $p(1) - p(0)$.

(D) ∞.

25. The function f defined by $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

(A) is differentiable for all real values of x.

(B) is not differentiable at $x = 0$.

(C) is not differentiable for $x < 0$.

(D) is not differentiable for $x > 0$.

26. Let $\{a_n\}$ be a sequence of distinct real numbers which has no convergent subsequence. Then $\lim_{n\to\infty} |a_n|$

(A) is 0.

(B) is ∞.

(C) is 1.

(D) does not exist.

27. The largest term in the sequence $x_n = \frac{1000^n}{n!}$, $n = 1, 2, 3, \ldots$

(A) is x_{999}.

(B) is x_{1001}.

(C) is x_1.

(D) does not exist.

28. A curve in $\mathbb{R}^2$ whose normal at each point passes through $(0, 0)$ is a

(A) straight line.

(B) parabola.

(C) hyperbola.

(D) circle.

29. Let f be a continuous function on $[0,1]$. Then $\lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{n} f(\frac{j}{n})$ is

(A) $\frac{1}{2}\int_0^{\frac{1}{2}} f(x)dx.$

(B) $\int_{\frac{1}{2}}^{1} f(x)dx.$

(C) $\int_0^1 f(x)dx.$

(D) $\int_0^{\frac{1}{2}} f(x)dx.$

30. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous. Let $D_u f(x,y)$ denote the directional derivative of f in the direction of $u \in \mathbb{R}^2$. If $D_{(1,1)}f(0,0) = 0$ and $D_{(1,-1)}f(0,0) = 0$, then

(A) $D_u f(0,0) = 1$ for some $u \in \mathbb{R}^2$.

(B) $D_u f(0,0) = -1$ for some $u \in \mathbb{R}^2$.

(C) $D_u f(0,0) = 0$ for all $u \in \mathbb{R}^2$.

(D) $D_u f(0,0)$ may not exist for some $u \in \mathbb{R}^2$.

31. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f \circ f = f$. Then

(A) f must be constant.

(B) $f(x) = x$ for all x in the range of f.

(C) f must be a non-constant polynomial.

(D) There is no such function.

32. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for $x \in [0,1]$.

If $f(x) \leq \int_0^x f(t)\,dt$ for all $0 \leq x \leq 1$, then

(A) $f(x) = 0$ for all $x \in [0,1]$.

(B) $f(x) = x$ for all $x \in [0,1]$.

(C) There is no such function.

(D) $f(x) = c$ for all $x \in [0,1]$ and some $c > 0$.

33. Consider the ordinary differential equation

$$y'' + 4y = \sin 2t, \quad y(0) = 0.$$

Then the solution $y(t)$

(A) converges to 0 as $t \to \infty$ with no oscillations.
(B) converges to 0 as $t \to \infty$ and the solution is oscillating.
(C) is oscillating and bounded.
(D) is unbounded.

34. Let $y(t)$ be a solution to the differential equation $y' = y^2 + t$, then $y(t)$ is differentiable

(A) once but not twice.
(B) twice but not 3 times.
(C) 3 times but not 4 times.
(D) infinitely many times.

35. Which of the following is a solution to the differential equation $y' = |y|^{1/2}$, $y(0) = 0$, where square root means the positive square root ?

(A) $y(t) = t^2/4$.
(B) $y(t) = -t^2/4$.
(C) $y(t) = t|t|/4$.
(D) $y(t) = -t|t|/4$.

36. The number of independent solutions of the differential equation $y^{(4)} - 2y^{(2)} + y = 0$ (here $y^{(2)}$ and $y^{(4)}$ represent the second and fourth derivatives of y respectively) is

(A) 4.
(B) 3.
(C) 2.
(D) 1.

37. The number of non-trivial polynomial solutions of the differential equation $x^3 y'(x) = y(x^2)$ is

(A) zero.

(B) one.

(C) three.

(D) infinity.

38. Let $\vec{p} = 3\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{q} = \vec{i} + 2\vec{j} + 3\vec{k}$ be vectors in $\mathbb{R}^3$ (here $\vec{i}, \vec{j}, \vec{k}$ denote the unit vectors along the positive X, Y, Z axes respectively). Suppose $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ is a unit vector such that $\vec{v} \cdot \vec{p} = 0 = \vec{v} \cdot \vec{q}$. The value of $|a + b + c|$ is :

(A) 6.

(B) 3.

(C) 1.

(D) 0.

39. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors in $\mathbb{R}^3$. If $\vec{a} \neq 0$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, then which of the following must certainly be true ?

(A) $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$

(B) $\vec{b} = \vec{c}$

(C) There is a real number λ such that $\vec{b} = \vec{c} + \lambda\,\vec{a}$

(D) $\vec{a}$ must be orthogonal to both $\vec{b}$ and $\vec{c}$

40. For a curve $\gamma : [a, b] \to \mathbb{R}^2$, let $\displaystyle\int_\gamma f$ denote the line integral of a function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ defined on some open set U containing $\{\gamma(t) : t \in [a, b]\}$. The value of $\displaystyle\int_{\mathbb{S}^1} f$, where $f(x, y) = \frac{y}{x^2+y^2}$ and $\mathbb{S}^1 = \{(\cos t, \sin t) : 0 \leq t \leq 2\pi\}$ (i.e, the circle of radius one centered at the origin) is

(A) 0.

(B) 1.

(C) π.

(D) 2π.