# Instructions

1. This question paper has forty multiple-choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble for a question, your answer for that question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively.

10. Let $[x]$ denote the greatest integer less than or equal to $x$ for a real number $x$.

# Integrated Ph. D./ Mathematical Sciences

1. The parametrized curve

$$x = \frac{1-u^2}{1+u^2}, y = \frac{2u}{1+u^2}, z = 2\tan^{-1} u, \quad -\infty < u < \infty$$

lies on a

(a) hyperboloid,
(b) sphere,
(c) cylinder,
(d) cone.

2. Let $0 < \alpha < \pi/4$ be an angle. The two straight lines $x^2 + y^2 = (x \tan \alpha - y)^2$ include an angle

(a) $2\alpha$,
(b) $\alpha$,
(c) $\pi/2 + 2\alpha$,
(d) $\pi/2 - 2\alpha$.

3. Let $P$ be a polynomial in $x$ such that

$$|P(x)| \leq C(1 + |x|)^{5/2} \text{ for all } x \in \mathbb{R}$$

and for some constant $C > 0$. Then

(a) $P$ is always linear,
(b) $P$ is of degree at most 2,
(c) $P$ is either a cubic polynomial or a quartic polynomial,
(d) $P(0)$ is always $C$.

4. Let $P, Q$ and $R$ be polynomials. Define

$$S(x) = P(x) + \frac{Q(x)}{1+x^2} + \frac{R(x)}{(1+x^2)^2}.$$

If $S(n) = 0$ for all $n \in \mathbb{Z}$, then

(a) $R(x)$ is always divisible by $1 + x^2$,

(b) $R(x)$ is always divisible by $(1 + x^2)^2$,

(c) $R(x)$ is always divisible by $(1 + x^2)^3$,

(d) $P, Q$ and $R$ must be identically zero.

5. Let $y(x)$ be a solution to the ordinary differential equation

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

with constants $A \neq 0, B \neq 0$ and $y(0) \neq 0$. Then

(a) $y(x)$ is always bounded on $\mathbb{R}$,

(b) $y(x)$ is always unbounded on $\mathbb{R}$,

(c) $y(x)$ is bounded or unbounded on $\mathbb{R}$ depending on the value of $y'(0)$,

(d) such a solution cannot exist.

6. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with $a, b, c$ and $d$ real. If $f'(x)$ has three distinct real roots, then

(a) there always exists a real constant $k$ such that $f(x) = k$ has exactly one real root,

(b) there always exists a real constant $k$ such that $f(x) = k$ has three distinct real roots,

(c) $f(x) = k$ has four distinct real roots for any real number $k$,

(d) $f(x) = k$ has two real and two complex roots for any real number $k$.

7. Let $X$ and $Y$ be non-empty sets and let

$$f_1, f_2 : X \longrightarrow Y \text{ and } g_1, g_2 : Y \longrightarrow X$$

be mappings. If $g_2 \circ f_2 \circ g_1 \circ f_1 : X \to X$ is a bijection, then

(a) $f_1, f_2, g_1$ must be injective,
(b) $f_2, g_1, g_2$ must be injective,
(c) $f_2, g_1$ must be bijective,
(d) $f_1$ must be injective and $g_2$ must be surjective.

8. A non-trivial binary relation $R$ on a non-empty set $X$ is called $(i)$ symmetric if $xRy$ implies $yRx$, $(ii)$ anti-symmetric if $xRy$ and $yRx$ imply $x = y$, $(iii)$ reflexive if $xRx$ holds for all $x$ in $X$, $(iv)$ transitive if $xRy$ and $yRz$ imply $xRz$, $(v)$ an equivalence relation if it is symmetric, reflexive and transitive. Moreover, $R$ is called the equality relation if $xRy$ holds if and only if $x = y$.

Let $\rho$ be a binary relation on a non-empty set $X$. If $\rho$ is symmetric and anti-symmetric, then

(a) $\rho$ is the equality relation,
(b) $\rho$ is an equivalence relation,
(c) $X$ has at most two elements,
(d) $\rho$ may not be reflexive.

9. Let $L$ be a line in the plane passing through the point $P = (4, 3)$. Let $Q$ and $R$ be the points of intersection of $L$ with the $x$-axis and the $y$-axis respectively. Let the segment $QR$ be in the first quadrant. If $PQ : PR = 5 : 3$, then the equation of $L$ is

(a) $9x + 20y = 96$,
(b) $\frac{x-4}{5} = \frac{y-3}{3}$,
(c) $\frac{x-4}{3} = \frac{y-5}{5}$,
(d) $3x + 5y = 27$.

10. Let $P(x) = x^4 + x^2 + 1$. Then $\int_{-1}^{1} P(\sin x) P'(\sin x) dx$ is

(a) 0,
(b) 1,
(c) $\pi$,
(d) $1/\pi$.

11. Consider
$$f(x) = |x|e^{ax} \text{ and } g(x) = xe^{ax}.$$
Then the pair $\{f, g\}$ is

(a) linearly independent on $\mathbb{R}$,
(b) linearly dependent on $(-\varepsilon, \varepsilon)$ for some $0 < \varepsilon < 1$,
(c) linearly independent on $(0, \infty)$,
(d) linearly independent on $(-\infty, 0)$.

12. Let $f : (-1, 1) \to \mathbb{R}$ be a function satisfying
$$\frac{1}{2}\,|x \log |x|| \le |f(x)| \le |x \log |x||$$
for all $x \neq 0$. Then

(a) $f$ is never differentiable at $x = 0$,
(b) $f$ is always differentiable at $x = 0$,
(c) $f$ is differentiable at $x = 0$ if $f(0) = 0$,
(d) $|f|$ is always differentiable at $x = 0$.

13. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to $A \neq 0$. Let $y_n = \max\{x_n, A - x_n\}$. Then the limit of the sequence $\{y_n\}_{n\in\mathbb{N}}$ is always

   (a) $0$,
   (b) $A$,
   (c) $\max\{A, 0\}$,
   (d) $-\min\{A, 0\}$.

14. Let

$$f(x) = \sqrt{x}\sin\frac{1}{x}$$

for $x > 0$. Set

$$x_n = f\left(\frac{2}{(2n+1)\pi}\right).$$

The sequence $\{x_n\}$

   (a) is a non-converging oscillatory sequence,
   (b) diverges to $\infty$,
   (c) converges to $\sqrt{\pi}$,
   (d) converges to 0.

15. Let the co-ordinates of the vertices of a plane triangle $\Delta$ be integers. If $a$ is the area of the triangle $\Delta$, then

   (a) $a$ must be an integer,
   (b) $a^2$ must be an integer,
   (c) $a + 1/2$ must be an integer,
   (d) $2a$ must be an integer.

16. Consider the second order ordinary differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where $b$ and $c$ are real constants. If

$$y = xe^{-5x}$$

is a solution, then

(a) both $b$ and $c$ are positive,
(b) $b$ is positive, but $c$ is negative,
(c) $b$ is negative, but $c$ is positive,
(d) both $b$ and $c$ are negative.

17. Let $x$ and $y$ be non-zero real numbers. Then the minimum value of

$$|x|^3 + |y|^3 + \frac{1}{|x|^3} + \frac{1}{|y|^3}$$

is

(a) 1/4,
(b) 1/2,
(c) 2,
(d) 4.

18. Consider the set $V = \{(x_1 - x_2 + x_3, x_1 + x_2 - x_3) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$. Then

(a) $V$ is not a vector subspace of $\mathbb{R}^2$,
(b) $V$ is a vector subspace of $\mathbb{R}^2$ of dimension 0 ,
(c) $V$ is a vector subspace of $\mathbb{R}^2$ of dimension 1,
(d) $V = \mathbb{R}^2$.

19. The set $S = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 + 6x + 4y + 12 = 0\}$ is

(a) an infinite set,
(b) a finite set with more than one element,
(c) a singleton,
(d) the empty set.

20. Suppose $f$ is a twice differentiable function defined on the real line with continuous second derivative. If $f$ satisfies $f(x+1) = f(x)$ for all $x \in \mathbb{R}$, then

$$\int_0^1 x f''(x) dx$$

is

(a) $f(0)$,
(b) $f(1)$,
(c) $f'(0)$,
(d) 0.

21. Let $f$ be a continuous function from $[0, 4]$ to $[3, 9]$. Then

(a) there must be an $x$ such that $f(x) = 4$,
(b) there must be an $x$ such that $3f(x) = 2x + 6$,
(c) there must be an $x$ such that $2f(x) = 3x + 6$,
(d) there must be an $x$ such that $f(x) = x$.

22. If $f$ is a differentiable function, define

$$(Df)(x) = \frac{1}{2x}\frac{df}{dx}.$$

Given $n \in \mathbb{N}$ and an $n$-times differentiable function $f$, define $(D^n f)(x)$ by successively applying $D$ for $n$ times to the function $f$. Then

$$D^8(e^{-x^2}) \text{ at } x = 0$$

is

(a) 1,
(b) 0,
(c) $e$,
(d) $e^{-1}$.

23. Let $A$ be a $5 \times 5$ matrix all of whose eigenvalues are zero. Then which of the following statements is always true?

(a) $A = -A$,
(b) $A^t = -A$,
(c) $A^t = A$,
(d) $A^5 = 0$.

24. If $x^2 - 2axy + y^2 - 2bx + 1 = 0$ represents a pair of straight lines, then the locus of the point $(a, b)$ is

(a) a circle,
(b) a straight line,
(c) a pair of straight lines,
(d) a point.

25. Consider the real four-dimensional vector space of all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. What is the dimension of the subspace of all those linear transformations that map the line $y = x$ into itself?

(a) 1,
(b) 2,
(c) 3,
(d) 4.

26. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $f(x) = 0$ for $|x| \geq 10$. Let

$$g(x) = \sum_{k \in \mathbb{Z}} f(x+k).$$

Then

(a) $g$ is differentiable and $g'$ has infinitely many zeros.

(b) $g$ is continuous and not differentiable,

(c) $g$ is differentiable and $g'$ has no zeros,

(d) $g$ is differentiable and $g'$ has only finitely many zeros.

27. For a positive integer $n$, let $I_n$ be the $n \times n$ identity matrix. Let $G$ be the function on positive integers defined by $G(n) =$ the number of $n \times n$ matrices $J$ with real entries which satisfy $J^2 + I_n = 0$. Then

(a) $G(n)$ can never be zero,

(b) $G(n) = 0$ if and only if $n$ is even,

(c) $G(n) = 0$ if and only if $n$ is odd,

(d) $G$ is identically equal to zero.

28. Let $g : \mathbb{R} \to (0, \infty)$ be an unbounded differentiable function and define

$$f(x) = \int_0^{g(x)} \sin^4(2t)\ dt.$$

Then which of the following statements is true?

(a) $f'$ is always positive,

(b) $f'$ has exactly one zero,

(c) $f'$ has exactly two distinct zeros,

(d) $f'$ has more than two distinct zeros.

29. The value of $\int_0^2 [x^2]dx$ lies between

(a) 1 and 1.5,
(b) 1.5 and 2,
(c) 2 and 2.5,
(d) 2.5 and 3.

30. Let $A$ be the group of rational numbers under addition. Let $M$ be the group of positive rational numbers under multiplication. Then which of the following is true?

(a) There are infinitely many isomorphisms between $A$ and $M$.
(b) There are only finitely many isomorphisms between $A$ and $M$.
(c) There is a unique isomorphism between $A$ and $M$.
(d) There is no isomorphism between $A$ and $M$.

31. Let $C$ be the subset of the right half plane contained within the lines $y = x$ and $y = -x$. Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points in $C$. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

(a) $|z_1 + z_2| \geq (1/\sqrt{2})(|z_1| + |z_2|)$ always,
(b) $|z_1 + z_2| < (1/\sqrt{2})(|z_1| + |z_2|)$ always,
(c) $|z_1 - z_2| \geq (1/\sqrt{2})(|z_1| + |z_2|)$ always,
(d) $|z_1 + z_2| = |z_1| + |z_2|$ always.

32. Let $G = \{a \in \mathbb{R} : a > 0, a \neq 1\}$. Then, under the binary operation $a * b = a^{\log b}$ the set $G$ is

(a) a commutative group,
(b) a non-commutative group,
(c) a semigroup but not a group,
(d) not a semigroup.

33. For each $t$ in $\mathbb{R}$, consider the straight lines $tax - by + t = 0$ and $ax + tby - 1 = 0$. Let $(u(t), v(t))$ be their point of intersection. Then the locus of the point $(u(t), v(t))$ is

(a) $a^2u^2 + b^2v^2 = 1$,
(b) $b^2u^2 + a^2v^2 = 1$,
(c) $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$,
(d) $u - v = 0$.

34. Let $f_1$ and $f_2$ be two real valued functions defined on the real line. Define two functions $g$ and $h$ by

$$g(x) = \max\{f_1(x), f_2(x)\} \text{ and } h(x) = \min\{f_1(x), f_2(x)\}.$$

Then

$$g(x)^2 + h(x)^2 + 3g(x)h(x) = f_1(x)^2 + f_2(x)^2 + 3f_1(x)f_2(x) \text{ holds for all } x \in \mathbb{R}$$

(a) always,
(b) only if $f_1(x) = f_2(x)$ for all $x$ in $\mathbb{R}$,
(c) only if $f_1$ and $f_2$ are both positive functions or both negative functions,
(d) only if at least one of the functions $f_1$ and $f_2$ is identically zero.

35. For each natural number $n \geq 2$, let $\mathbb{Z}_n$ denote the additive group of integers modulo $n$. Suppose $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{mn}$. Then

(a) $m$ divides $n$ or $n$ divides $m$,
(b) $m = n$,
(c) $m + n = mn$,
(d) gcd $(m, n) = 1$.

36. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a three times differentiable map. Moreover, suppose $\gamma'(t)$ is a unit vector for every $t \in \mathbb{R}$. Then

(a) $\gamma'(t)$ is orthogonal to $\gamma''(t)$ for every $t \in \mathbb{R}$,
(b) $\gamma'(t)$ is parallel to $\gamma''(t)$ for every $t \in \mathbb{R}$,
(c) the image of $\gamma$ must lie on a sphere,
(d) the image of $\gamma$ must lie in a plane.

37. Given a group $G$, its center is

$$\{h \in G : hg = gh \text{ for all } g \in G\}.$$

Let $G$ be the group

$$G = \{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}\}.$$

Then the center of $G$

(a) is $G$ itself,
(b) is $\{\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{Z}\}$.
(c) consists only of the $3 \times 3$ identity matrix,
(d) $\{\begin{pmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{Z}\}$.

38. Let $G$ be a group of order 6. The number of elements of order 2 in the group $G$ is

(a) exactly 1,
(b) exactly 2,
(c) exactly 3,
(d) either 1 or 3.

39. Let $\mathbb{F}_2 = \{0, 1\}$ be the field with two elements. Consider

$$GL_2(\mathbb{F}_2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_2 \text{ and } ad \neq bc\}$$

with the usual matrix multiplication. Then $GL_2(\mathbb{F}_2)$ is

(a) a commutative group,

(b) a non-commutative group,

(c) a semi-group, but not a group,

(d) not a semi-group.

40. Let **a**, **b**, **c** be three vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq 0$. Then which of the following is true:

(a) **c** is parallel to **a**,

(b) **b** is a linear combination of **a** and **c**,

(c) **b** is orthogonal to both **a** and **c**,

(d) **c** is orthogonal to **a**.