Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively.
10. Let $[x]$ denote the greatest integer less than or equal to x for a real number x.

Integrated Ph. D. Mathematical Sciences

1. Let A be an $n \times n$ matrix with real entries such that $A^2 + I = 0$. Then

   (A) n is an odd integer.
   (B) n is an even integer.
   (C) n has to be 2.
   (D) n could be any positive integer.

2. Consider the group

$$G = \left\{ \begin{pmatrix} \lambda & a \\ 0 & \mu \end{pmatrix} : a \in \mathbb{C} \text{ and } \lambda, \mu \in \mathbb{C} \setminus \{0\} \right\}.$$

   Then the subset

$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & \mu \end{pmatrix} : a \in \mathbb{C} \text{ and } \mu \in \mathbb{C} \setminus \{0\} \right\}$$

   is

   (A) a normal subgroup
   (B) a subgroup but not a normal subgroup.
   (C) not a subgroup in general.
   (D) an abelian subgroup.

3. Let k be a positive integer. Let $n_1, n_2, \ldots, n_k$ and n be integers, each greater than one. Suppose they satisfy

$$\sum_{i=1}^{k} (1 - \frac{1}{n_i}) = 2 - \frac{2}{n}.$$

   Then the only possible values of k are

(A) any integer.

(B) 1 and 2.

(C) 2 and 3.

(D) 3 and 4.

4. Let S_4 be the group of all permutations of 4 symbols. Let H be the following subset of S_4:

$$H = \{e, (12)(34), (13)(24), (14)(23)\},$$

where e stands for the identity permutation. Then

(A) H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(B) H is isomorphic to $\mathbb{Z}_4$.

(C) H is not a subgroup.

(D) H is isomorphic to $\mathbb{A}_4$.

5. Suppose f is a continuous real-valued function. Let $I = \int_0^1 f(x)x^2 dx$. Then it is necessarily true that I equals

(A) $\frac{f(1)}{3} - \frac{f(0)}{3}$.

(B) $\frac{f(c)}{3}$ for some $c \in [0, 1]$.

(C) $f(\frac{1}{3}) - f(0)$.

(D) $f(c)$ for some $c \in [0, 1]$.

6. Let G be a finite group of odd order. Let $f : G \to G$ be the function defined by $f(g) = g^2$. Then f is

(A) always an isomorphism.

(B) always a bijection, but not necessarily an isomorphism.

(C) never an isomorphism.

(D) not always a bijection.

7. If V is a ten dimensional vector space, then the dimension of the intersection of two six dimensional subspaces

(A) is always 6.

(B) can be any integer between 0 and 6, both inclusive.

(C) can be any integer between 2 and 6, both inclusive.

(D) can be any integer between 4 and 6, both inclusive.

8. Let S_3 denote the permutation group on 3 symbols and let $\mathbb{R}^*$ denote the multiplicative group of non-zero real numbers. Suppose

$$h : S_3 \to \mathbb{R}^*$$

is a homomorphism. Then kernel of h has

(A) always at most 2 elements.
(B) always at most 3 elements.
(C) always at least 3 elements.
(D) always exactly 6 elements.

9. Let $y(x)$ be a solution of the ODE

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + By = 0,$$

where $0 < B < 1$. Then $\lim_{x\to\infty} y(x)$ equals

(A) 0.
(B) $+\infty$.
(C) $-\infty$.
(D) $B/2$.

10. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{(n+1)^{3/2}} + \cdots + \frac{1}{(2n)^{3/2}}.$$

As $n \to \infty$, the sequence a_n

(A) converges to 0.
(B) diverges to ∞.
(C) is bounded but does not converge.
(D) converges to a positive number.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable even function. Let

$$G(x) = \int_0^{f(x)} \sqrt{\tan\theta}\, d\theta.$$

Then the value of $G'(0)$

(A) equals -1.

(B) equals 0.

(C) equals 1.

(D) cannot be determined from the given data.

12. Let X be a non-empty set and let $f, g : X \to X$ be functions. Suppose $f \circ g \circ f$ equals the identity function on X. Then

(A) g is one-one but not necessarily onto.

(B) g is onto but not necessarily one-one.

(C) g is one-one and onto.

(D) g is necessarily the identity function on X.

13. Let G be the additive group of integers modulo 12. The number of different isomorphisms of G onto itself is

(A) 3.

(B) 4.

(C) 12.

(D) 24.

14. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be

$$f(x, y, z) = xye^{-z} - xze^{-y} + yze^{-x}.$$

The unit vector $\mathbf{u}$ that maximizes the directional derivative of f in the direction of $\mathbf{u}$ at the point $(1, 0, 0)$ is

(A) $\frac{1}{\sqrt{2}}(1, -1, 0)$.

(B) $\frac{1}{\sqrt{2}}(0, 1, -1)$.

(C) $\frac{1}{\sqrt{2}}(-1, 0, 1)$.

(D) $\frac{1}{\sqrt{3}}(1, -1, 1)$.

15. Consider the second order ODE

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + B = 0$$

where A and B are positive real numbers. The equation

(A) always admits a linearly independent pair of solutions that are trigonometric functions.

(B) always admits a linearly independent pair of solutions that are products of exponential and trigonometric functions.

(C) need not admit a linearly independent pair of solutions that are products of exponential and trigonometric functions.

(D) need not admit any solution.

16. Consider the following subsets of $\mathbb{R}^3$:

$$X_1 = \{(x, y, z) \in \mathbb{R}^3 : z^2 - x^2 + 16x - y^2 + 9y = 25\},$$
$$X_2 = \{(x, y, z) \in \mathbb{R}^3 : x + z = 9\}.$$

Then $X_1 \cap X_2$ is

(A) a pair of lines,

(B) an ellipse lying in some plane in $\mathbb{R}^3$,

(C) a parabola lying in some plane in $\mathbb{R}^3$,

(D) a hyperbola lying in some plane in $\mathbb{R}^3$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(f(x)) = x \text{ for all } x.$$

Then

(A) f is monotone.

(B) f has to be the identity.

(C) f need not be monotone.

(D) $f(x) = \sqrt{x}$.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x) - f(y)| \leq |x - y|^2 \text{ for all } x, y \in \mathbb{R}.$$

Then

(A) f has to be a linear function.

(B) $f(x) = x^2$.

(C) f has to be a constant.

(D) f has to be the identity function.

19. Let A be an $n \times n$ real non-zero matrix of rank less than n. Then

(A) there exists an $n \times n$ real non-zero matrix B such that $BA = 0$.

(B) there may not always exist an $n \times n$ real non-zero matrix B such that $BA = 0$.

(C) there exists an $n \times n$ real non-zero matrix B such that $BA = I$.

(D) if B is such that $BA = 0$, then $AB = 0$.

20. Let T be a 4×4 matrix with real entries. Suppose $T^5 = 0$. Then which of the following is necessarily true?

(A) T is the zero matrix.

(B) T need not be the zero matrix, but T^2 is the zero matrix.

(C) T^2 need not be the zero matrix, but T^3 is the zero matrix.

(D) T^3 need not be the zero matrix, but T^4 is the zero matrix.

21. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $f(0) = 0$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Then there exists c in $\mathbb{R}$ such that

(A) $|f(x)| \leq c\sqrt{|x|}$ for all x with $|x| \geq 1$.

(B) $|f(x)| \leq c|x|^2$ for all x with $|x| \geq 1$.

(C) $f(x) = x + c$ for all $x \in \mathbb{R}$.

(D) $f(x) = 0$ for all $x \in \mathbb{R}$.

22. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that $\nabla f(x) \times \mathbf{u} = 0$ for all $x \in \mathbb{R}^3$ where $\mathbf{u}$ is the vector $(1, 0, 0)$. Then it must be that

(A) $f(x_1, y_1, z) = f(x_2, y_2, z)$ for all x_1, y_1, x_2, y_2 and z.

(B) $f(x_1, y, z_1) = f(x_2, y, z_2)$ for all x_1, z_1, x_2, z_2 and y.

(C) $f(x, y_1, z_1) = f(x, y_2, z_2)$ for all y_1, z_1, y_2, z_2 and x.

(D) f is a constant function.

23. Let l be a line segment realizing the distance between a circle C and an ellipse E in the plane. Then

(A) l must meet C orthogonally, but need not meet E orthogonally.

(B) l need not meet C or E orthogonally.

(C) l must meet E orthogonally, but need not meet C orthogonally.

(D) l must meet both C and E orthogonally.

24. Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^3$. Then,

(A) there is a unique $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$.

(B) there is a $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$, but this need not be unique.

(C) there may not exist any $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$.

(D) there is a unit vector $\mathbf{y}$ in $\mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{y} = \mathbf{v}$ if and only if $\|\mathbf{u}\| \geq \|\mathbf{v}\|$.

25. Let $f : G_1 \to G_2$ be a homomorphism of the group G_1 into the group G_2. Let H be a subgroup of G_2. Then which of the following is true?

(A) If H is abelian, then $f^{-1}(H)$ is an abelian subgroup of G_1.

(B) If H is normal, then $f^{-1}(H)$ is a normal subgroup of G_1.

(C) $f^{-1}(H)$ need not be a subgroup of G_1.

(D) $f^{-1}(H)$ must be contained in the kernel of f.

26. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a non-zero linear transformation such that $T\mathbf{v} = 0$ for all $\mathbf{v} \in S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, x + y + z = 0\}$. Then the dimension of the kernel of T has to be

(A) 0.

(B) 1.

(C) 2.

(D) 0 or 1.

27. Let R be the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. Consider R with usual addition and multiplication of matrices. Which of the following is true?

(A) R is a ring without zero-divisors.

(B) R is a ring with zero-divisors.

(C) R is a commutative ring.

(D) Every non-zero element in R has a multiplicative inverse.

28. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-coplanar unit vectors such that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \frac{\mathbf{v} + \mathbf{w}}{2}.$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be α and let the angle between $\mathbf{u}$ and $\mathbf{w}$ be β with $0 \leq \alpha, \beta \leq \pi$. Then (α, β) equals

(A) $(\frac{\pi}{3}, \frac{\pi}{3})$.

(B) $(\frac{2\pi}{3}, \frac{\pi}{3})$.

(C) $(\frac{\pi}{3}, \frac{2\pi}{3})$.

(D) $(\frac{2\pi}{3}, \frac{2\pi}{3})$.

29. Let

$$\mathbf{u} = \mathbf{i} + 4x\mathbf{j} + (x - 6)\mathbf{k} \text{ and } \mathbf{v} = y^2\mathbf{i} + y\mathbf{j} + 4\mathbf{k}$$

where $x, y \in \mathbb{R}$. If the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute for all $y \in \mathbb{R}$, then

(A) $x < 2$.

(B) $x > 3$.

(C) $x < -3$ or $x > 2$.

(D) $-2 < x < 3$.

30. Consider the two parabolas $y^2 = 4ax$ and $x^2 = 4by$. Suppose, given any point in the plane, the tangents to the first parabola from that point are normal to the second. Then

(A) $a = \pm b$.

(B) $ab = 4$.

(C) $a^2 > 8b^2$.

(D) $a^2 < 8b^2$.

31. Let α, β, a and b be real constants with a and b non-zero. Let θ be a real number. Let $P(\theta) = (a\tan(\theta + \alpha), b\tan(\theta + \beta))$. Then $P(\theta)$ lies on

(A) a hyperbola.

(B) a parabola.

(C) an ellipse.

(D) a straight line.

32. A tangent to the parabola $x^2 = 4y$ meets the hyperbola $xy = 1$ in P and Q. If the tangent varies, then the locus of the mid-point of P and Q is

(A) a straight line,

(B) a hyperbola,

(C) an ellipse,

(D) a parabola.

33. Let A be a subset of $\mathbb{R}^3$ such that

$$tx + (1-t)y \in A$$

for all x and y in A and for all t in $\mathbb{R}$. Then

(A) the set A is a straight line,

(B) for any $u \in A$, the set $A_u = \{v - u : v \in A\}$ is a vector subspace of $\mathbb{R}^3$,

(C) the set A is a vector subspace of $\mathbb{R}^3$,

(D) the set A is a bounded convex set.

34. Which of the following need not be true for an $n \times n$ real matrix A?

(A) If columns of A span $\mathbb{R}^n$, then rows of A span $\mathbb{R}^n$.

(B) If columns of A are linearly independent, then rows of A are linearly independent.

(C) If columns of A are orthogonal, then rows of A are orthogonal.

(D) If columns of A are orthonormal, then rows of A are orthonormal.

35. Let $\mathcal{S}$ be a collection of non-empty subsets of $\{1, 2, \ldots, 10\}$ such that if $A, B \in \mathcal{S}$, then either $A \subset B$ or $B \subset A$. The maximum possible cardinality of $\mathcal{S}$ is

(A) 10.

(B) $\binom{10}{2}$.

(C) $\binom{10}{5}$.

(D) $\binom{10}{6}$.

36. The value of

$$\lim_{n\to\infty}(1+\frac{1}{n})^{n^2}e^{-n}$$

is

(A) 1.

(B) $e^{-1/2}$.

(C) e.

(D) e^2.

37. The value of

$$\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\frac{1}{4.5}+\cdots$$

is

(A) between 0 and 1/4.

(B) between 1/4 and 1/3.

(C) between 1/3 and 1/2.

(D) between 1/2 and 1.

38. Which of the following is true?

(A) $e^{\frac{1}{2}(x+y)} \geq \frac{1}{2}(e^x + e^y)$,

(B) $\log\frac{x+y}{2} \geq \frac{1}{2}(\log x + \log y)$,

(C) $\frac{1}{2}(x^{\frac{3}{2}} + y^{\frac{3}{2}}) \geq \left(\frac{x+y}{2}\right)^{3/2}$,

(D) $\frac{1}{2}(xe^{-x} + ye^{-y}) \leq \frac{1}{2}((x+y)e^{-\frac{x+y}{2}})$.

39. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, which of the statements implies that $f(0) = 0$?

(A) $\int_0^1 f(x)^n dx \to 0$ as $n \to \infty$.

(B) $\int_0^1 f(\frac{x}{n}) dx \to 0$ as $n \to \infty$.

(C) $\int_0^1 f(nx) dx \to 0$ as $n \to \infty$.

(D) $\int_0^1 f(x+n) dx \to 0$ as $n \to \infty$.

40. The image of a circle under a non-constant linear transformation can be

(A) a rectangle,

(B) a parabola,

(C) an ellipse,

(D) Any of the above.