Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded **2.5** marks, but **0.5** marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively.

10. Let S_n denote the group of permutations of $\{1, 2, \cdots, n\}$ and $\mathbb{Z}_n$ the group $\mathbb{Z}/n\mathbb{Z}$.

11. Let $f : X \to Y$ be a function. For $A \subset X$, $f(A)$ denotes the image of A under f.

Integrated Ph. D. Mathematical Sciences

1. Consider the following system of linear equations.

$$\begin{aligned} x + y + z + w &= b_1. \\ x - y + 2z + 3w &= b_2. \\ x - 3y + 3z + 5w &= b_3. \\ x + 3y - w &= b_4. \end{aligned}$$

   For which of the following choices of b_1, b_2, b_3, b_4 does the above system have a solution?

   (A) $b_1 = 1, b_2 = 0, b_3 = -1, b_4 = 2.$

   (B) $b_1 = 2, b_2 = 3, b_3 = 5, b_4 = -1.$

   (A) $b_1 = 2, b_2 = 2, b_3 = 3, b_4 = 0.$

   (A) $b_1 = 2, b_2 = -1, b_3 = -3, b_4 = 3.$

2. Let $y : [0,1] \to \mathbb{R}$ be a twice continuously differentiable function such that,

$$\frac{d^2y}{dx^2}(x) - y(x) < 0, \text{ for all } x \in (0,1), \text{and } y(0) = y(1) = 0.$$

   Then,

   (A) y has at least two zeros in $(0,1)$.

   (B) y has at least on zero in $(0,1)$.

   (C) $y(x) > 0$ for all $x \in (0,1)$.

   (D) $y(x) < 0$ for all $x \in (0,1)$.

3. Which one of the following boundary value problems has more than one solution?

   (A) $y'' + y = 1,\ y(0) = 1,\ y(\pi/2) = 0.$

   (B) $y'' + y = 1,\ y(0) = 0,\ y(2\pi) = 0.$

   (C) $y'' - y = 1,\ y(0) = 0,\ y(\pi/2) = 0.$

   (D) $y'' - y = 1,\ y(0) = 0,\ y(\pi) = 0.$

4. Let A be an $n \times n$ nonsingular matrix such that the elements of A and A^{-1} are all integers. Then,

(A) $\det A$ must be a positive integer.

(B) $\det A$ must be a negative integer.

(C) $\det A$ can be $+1$ or -1.

(D) $\det A$ must be $+1$.

5. Let Q be a polynomial of degree 23 such that $Q(x) = -Q(-x)$ for all $x \in \mathbb{R}$ with $|x| \geq 10$. If $\int_{-1}^{1} (Q(x) + c)\, dx = 4$ then c equals

(A) 0.

(B) 1.

(C) 2.

(D) 4.

6. Let $b > 0$ and $x_1 > 0$ be real numbers. Then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{b}{x_n}\right)$$

(A) diverges.

(B) converges to $\sqrt{x_1}$.

(C) converges to $\sqrt{(b + x_1)}$.

(D) converges to $\sqrt{b}$.

7. Let $f(x) = \begin{cases} \frac{3x}{4} & \text{if } x \in \mathbb{Q}. \\ \sin x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Then the number of points where f is continuous equals

(A) 1.

(B) 2.

(C) 3.

(D) ∞.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying, $f(x) = 5\int_0^x f(t)\, dt + 1, \quad \forall x \in \mathbb{R}$. Then $f(1)$ equals

(A) e^5.

(B) 5.

(C) $5e$.

(D) 1.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g(x) = \int_0^{x^2+3x+2} f(t)\, dt$. Then, $g'(0)$ equals

   (A) $3f(2)$.
   (B) $f(2)$.
   (C) $3f(0)$.
   (D) $f(0)$.

10. Let $x_n > 0$ be such that $\sum_{n=1}^{\infty} x_n$ diverges and $\sum_{n=1}^{\infty} x_n^2$ converges. Then x_n **cannot** be

   (A) $\frac{n}{n^2+1}$.
   (B) $\frac{\log n}{n}$.
   (C) $\frac{1}{n\sqrt{\log n}}$.
   (D) $\frac{1}{n(\log n)^2}$.

11. If B is a subset of $\mathbb{R}^3$ and $u \in \mathbb{R}^3$, define $B - u = \{w - u : \ w \in B\}$. Let $A \subset \mathbb{R}^3$, be such that $tu + (1-t)v \in A$ whenever $u, v \in A$ and $t \in \mathbb{R}$. Then,

   (A) A must be a straight line.
   (B) A must be a line segment.
   (C) $A - u_0$ is a subspace for a unique $u_0 \in A$.
   (D) $A - u$ is a subspace for all $u \in A$.

12. Minimum value of $|z+1| + |z-1| + |z-i|$ for $z \in \mathbb{C}$ is

   (A) 2.
   (B) $2\sqrt{2}$.
   (C) $1 + \sqrt{3}$.
   (D) $\sqrt{5}$.

13. The minimum value of $|z-w|$ where $z, w \in \mathbb{C}$ such that $|z| = 11$, and $|w+4+3i| = 5$ is

   (A) 1.
   (B) 2.
   (C) 5.

(D) 6.

14. Let $\mathcal{P}$ be the vector space of polynomials with real coefficients. Let T and S be two linear maps from $\mathcal{P}$ to itself such that $T \circ S$ is the identity map. Then,

    (A) $S \circ T$ may not be the identity map.
    (B) $S \circ T$ must be the identity map, but T and S need not be the identity maps.
    (C) T and S must both be the identity map.
    (D) There is a scalar α such that $T(p) = \alpha p$ for all $p \in \mathcal{P}$.

15. Let ℓ_1 and ℓ_2 be two perpendicular lines in $\mathbb{R}^2$. Let P be a point such that the sum of the distances of P from ℓ_1 and ℓ_2 equals 1. Then the locus of P is

    (A) a square.
    (B) a circle.
    (C) a straight line.
    (D) a set of four points.

16. Let $0 < b < a$. A line segment AB of length b moves on the plane such that A lies on the circle $x^2 + y^2 = a^2$. Then the locus of B is

    (A) a circle.
    (B) union of two circles.
    (C) a region bounded by two concentric circles.
    (D) an ellipse, but not a circle.

17. Let u, v and w be three vectors in $\mathbb{R}^3$. It is given that $u \cdot u = 4$, $v \cdot v = 9$, $w \cdot w = 1$, $u \cdot v = 6$, $u \cdot w = 0$ and $v \cdot w = 0$. Then the dimension of the subspace spanned by $\{u, v, w\}$ is

    (A) 1.
    (B) 2.
    (C) 3.
    (D) cannot be determined.

18. Let a_n be the number of ways of arranging n identical black balls and $2n$ identical white balls in a line so that no two black balls are next to each other. Then a_n equals

    (A) $3n$.

(B) $\binom{2n+1}{n}$.

(C) $\binom{2n}{n}$.

(D) $\binom{2n-1}{n(2n+1)}$.

19. Maximal area of a triangle whose vertices are on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(A) $\frac{3\sqrt{3}}{4}ab$.

(B) $\frac{3\sqrt{3}}{4}\frac{(a^2+b^2)}{2}$.

(C) $\frac{3\sqrt{3}}{4}\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}$.

(D) $\frac{3\sqrt{3}}{4}$.

20. Let $a_k=\frac{1}{2^{2k}}\binom{2k}{k}$, $k=1,2,3,\cdots$. Then

(A) a_k is increasing.

(B) a_k is decreasing.

(C) a_k decreases for first few terms and then increases.

(D) none of the above.

21. What is the limit of $(2^n+3^n+4^n)^{\frac{1}{n}}$ as $n\to\infty$?

(A) 0.

(B) 1.

(C) 3.

(D) 4.

22. What is the limit of $e^{-2n}\sum_{k=0}^{n}\frac{(2n)^k}{k!}$ as $n\to\infty$?

(A) 0.

(B) 1.

(C) $1/e$.

(D) e.

23. Let $f,g:[-1,1]\to\mathbb{R}$ be **odd** functions whose derivatives are continuous. You are given that $|g(x)|<1$ for all $x\in[-1,1]$, $f(-1)=-1$, $f(1)=1$ and that $f'(0)<g'(0)$. Then the minimum possible number of solutions to the equation $f(x)=g(x)$ in the interval $[-1,1]$ is

(A) 1.

(B) 3.

(C) 5.

(D) 7.

24. Let $f : S_3 \to \mathbb{Z}_6$ be a group homomorphism. Then the number of elements in $f(S_3)$ is

(A) 1.

(B) 1 or 2.

(C) 1 or 3.

(D) 1 or 2 or 3.

25. Consider the multiplicative group $S = \{z : \ |z| = 1\} \subset \mathbb{C}$. Let G and H be subgroups of order 8 and 10 respectively. If n is the order of $G \cap H$ then

(A) $n = 1$.

(B) $n = 2$.

(C) $3 \leq n \leq 5$.

(D) $n \geq 6$.

26. Let G be a finite abelian group. Let H_1 and H_2 be two distinct subgroups of G of index 3 each. Then the index of $H_1 \cap H_2$ in G is

(A) 3.

(B) 6.

(C) 9.

(D) Cannot be computed from the given data.

27. A particle follows the path $c : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^3$, $c(t) = (\cos t, 0, |\sin t|)$. Then the distance travelled by the particle is

(A) $\frac{3\pi}{2}$.

(B) π.

(C) 2π.

(D) 1.

28. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the map given by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2, x_2 - 2x_3, x_3 - 2x_1, x_1 - 2x_3).$$

Then the dimension of $T(\mathbb{R}^3)$ equals

(A) 1.
(B) 2.
(C) 3.
(D) 4.

29. The tangent plane to the surface $z^2 - x^2 + \sin(y^2) = 0$ at $(1, 0, -1)$ is

(A) $x - y + z = 0.$
(B) $x + 2y + z = 0.$
(C) $x + y - 1 = 0.$
(D) $x + z = 0.$

30. Let A and B be two 3×3 matrices with real entries such that $\mathrm{rank}(A) = \mathrm{rank}(B) = 1$. Let $N(A)$ and $R(A)$ stand for the null space and range space of A. Define $N(B)$ and $R(B)$ similarly. Then which of the following is necessarily true ?

(A) $\dim(N(A) \cap N(B)) \geq 1.$
(B) $\dim(N(A) \cap R(A)) \geq 1.$
(C) $\dim(R(A) \cap R(B)) \geq 1.$
(D) $\dim(N(A) \cap R(A)) \geq 1.$

31. For a permutation π of $\{1, 2, \cdots, n\}$, we say that k is a fixed point if $\pi(k) = k$. Number of permutations in S_5 having exactly one fixed point is

(A) 24.
(B) 45.
(C) 60.
(D) 96.

32. Let $A = \{1, 2, \cdots, 10\}$. If S is a subset of A, let $|S|$ denote the number of elements in S. Then

$$\sum_{S \subset A, S \neq \phi} (-1)^{|S|}$$

equals

(A) -1.

(B) 0.

(C) 1.

(D) 10.

33. Let $\mathcal{P}_m$ be the vector space of polynomials with real coefficients of degree less than or equal to m. Define $T : \mathcal{P}_m \to \mathcal{P}_m$ by $T(f) = f' + f$. Then the dimension of range(T) equals

(A) 1

(B) $(m-1)$.

(C) m.

(D) $(m+1)$.

34. Let A and B be two finite sets of cardinality 5 and 3 respectively. Let G be the collection of all mappings f from A into B such that the cardinality of $f(A)$ is 2. Then, cardinality of G equals

(A) $3 \cdot 2^5 - 6$.

(B) $3 \cdot 2^5$.

(C) $3 \cdot 5^2$.

(D) $\frac{1}{2}(3^5 - 3)$.

35. Let G be the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and let H be the collection of all isomorphisms from G onto itself. Then the cardinality of H is

(A) 2.

(B) 4.

(C) 6.

(D) 8.

36. A line L in the XY-plane has intercepts a and b on X-axis and Y-axis respectively. When the axes are rotated through an angle θ (keeping the origin fixed), L makes equal intercepts with the axes. Then $\tan\theta$ equals

(A) $\frac{a-b}{a+b}$.

(B) $\frac{a-b}{2(a+b)}$.

(C) $\frac{a+b}{a-b}$.

(D) $\frac{a^2-b^2}{a^2+b^2}$.

37. Let B_1, B_2 and B_3 be three distinct points on the parabola $y^2 = 4x$. The tangents at B_1, B_2 and B_3 to the parabola (taken in pairs) intersect at C_1, C_2 and C_3. If a and A are the areas of the triangles $B_1B_2B_3$ and $C_1C_2C_3$ respectively, then

(A) $a = A$.

(B) $a = 2A$.

(C) $2a = A$.

(D) $a = \sqrt{2}A$.

38. Let P be a 3×2 matrix, Q be a 2×2 matrix and R be a 2×3 matrix such that PQR is equal to the identity matrix. Then,

(A) rank of $P = 2$.

(B) Q is nonsingular.

(C) Both (A) and (B) are true.

(D) There are no such matrices P, Q and R.

39. The number of elements of order 3 in the group $\mathbb{Z}_{15} \times \mathbb{Z}_{15}$ is

(A) 3.

(B) 8.

(C) 9.

(D) 15.

40. The number of surjective group homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_3$ equals

(A) 1.

(B) 2.

(C) 3.

(D) ∞.