INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2009

Program : Research

Entrance Paper : Mathematics

Paper Code : MA

Day & Date
SUNDAY, 26TH APRIL 2009

Time
9.00 A.M. TO 12.00 NOON

Instructions

(1) This question paper consists of two parts: Part A and Part B and carries a total of 100 Marks.

(2) There is no negative marking.

(3) Candidates are asked to fill in the required fields on the sheet attached to the answer book.

(4) Part A carries 20 multiple choice questions of 2 marks each. Answer all questions in Part A.

(5) Answers to Part A are to be marked in the OMR sheet provided.

(6) For each question, darken the appropriate bubble to indicate your answer.

(7) Use only HB pencils for bubbling answers.

(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

(10) Part B has 8 questions. Answer any 6 in this part. Each question in this part carries 10 marks.

(11) Answers to Part B are to be written in the separate answer book provided.

(12) **Answer to each question in Part B should begin on a new page.**

(13) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ ($\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $\mathbb{C}_+$) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.

(14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denote the standard norm on $\mathbb{R}^n$. The metric given by $d(x,y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

MATHEMATICS

PART A

(1) Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Assume that $\int_{-1}^{1} f(t)dt = 2$. Then

$$\lim_{n\to\infty} \int_{-1}^{1} f(t) sin^2(nt) dt$$

(A) equals 0.

(B) equals 1.

(C) equals $f(1) - f(-1)$.

(D) does not exist.

(2) The radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \left(\frac{a^n}{n} + \frac{b^n}{n^2} \right) x^n$$

where $a > 0$, $b > 0$ and $a \neq b$, is

(A) $R = 0$.

(B) $R = \infty$.

(C) $R = min(1/a. 1/b)$.

(D) $R = max(1/a, 1/b)$.

(3) For $z = x + iy \in \mathbb{C}$,

$$\left| e^{z^2} \right| = e^{|z|^2}$$

holds

(A) for all $z \in \mathbb{C}$.

(B) if and only if $y = 0$.

(C) if and only if $x = 0$.

(D) only when $z = 0$.

(4) Let C be the circle $\{|z| = 1\}$ in the complex plane described counterclockwise. Then

$$\int_C \frac{1+z}{(2-z)z} dz$$

equals

(A) πi.

(B) $-\pi i$.

(C) $2\pi i$.

(D) $-2\pi i$.

(5) Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has left and right derivatives at 0. Then, at $x = 0$,

(A) f must be continuous but may not be differentiable.

(B) f need not be continuous but must be left continuous or right continuous.

(C) f must be differentiable.

(D) If f is continuous then f must be differentiable.

(6) Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Suppose that for each $\epsilon > 0$, there is a subsequence $\{x_{n_k}\}_{k\geq 1}$ so that $x_{n_k} \leq x + \epsilon$, for all $k \geq 1$. Then we must have

(A) $\limsup_{n\to\infty} x_n \leq x$.

(B) $\limsup_{n\to\infty} x_n \geq x$.

(C) $\liminf_{n\to\infty} x_n \leq x$.

(D) $\liminf_{n\to\infty} x_n \geq x$.

(7) Let the sequence of functions $f_n : [0,1] \to \mathbb{R}$, $n \geq 2$, be given by

$$f_n = \begin{cases} n^2 x, & 0 \leq x \leq 1/n, \\ 2n - n^2 x, & 1/n < x < 2/n, \\ 0, & 2/n \leq x \leq 1. \end{cases}$$

Then,

(A) f_n converges pointwise but not uniformly as $n \to \infty$.

(B) f_n converges uniformly as $n \to \infty$.

(C) The functions $\{f_n\}_{n\geq 1}$ are equicontinuous.

(D) $\int_0^1 f_n(x)dx$ converges to 0 as $n \to \infty$.

(8) The function $f(x) = e^{-|x|}$ is

(A) continuous but not uniformly continuous.

(B) uniformly continuous but not differentiable.

(C) differentiable but not uniformly continuous.

(D) differentiable and uniformly continuous.

(9) Let A_n be the sequence of intervals

$$A_n = \left(1 + \frac{(-1)^n}{n}, 1 + \frac{2}{n}\right)$$

for $n \geq 1$. Then

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_n)$$

is

(A) the empty set.

(B) $\{1\}$.

(C) $(0, 3)$.

(D) $(0, 1)$.

(10) Let $\mathbf{v}$ and $\mathbf{w}$ be 3×1 row vectors. If $\mathbf{w}^T$ denotes the transpose of $\mathbf{w}$, then for the matrix $\mathbf{v}\mathbf{w}^T$

(A) 0 is not an eigenvalue.

(B) 0 is an eigenvalue with multiplicity 1.

(C) 0 is an eigenvalue with multiplicity 2.

(D) 0 is an eigenvalue with multiplicity 3.

(11) Let A be the matrix

$$A = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix}$$

with $a, c \in \mathbb{R}$ and $c \neq 0$. Then there is a 2×2 matrix P such that PAP^{-1} is diagonal

(A) for all values of a.

(B) for no value of a.

(C) if and only if $a = c$.

(D) if and only if $a = 0$.

(12) Let A be a 3×3 matrix over $\mathbb{R}$ such that $AB = BA$ for all 3×3 matrices B over $\mathbb{R}$. Then

(A) A must be I or 0.

(B) A must be diagonal.

(C) A must be orthogonal.

(D) A must have 3 distinct eigenvalues.

(13) Let $V, W \subset \mathbb{R}^5$ be subspaces with $dim(V) = dim(W) = 3$. Let

$$V + W = \{v + w : v \in V, w \in W\}$$

(A) We always have $V + W = \mathbb{R}^5$.

(B) We never have $V + W = \mathbb{R}^5$.

(C) We must have $\dim(V \cap W) \geq 1$.

(D) If $V + W = \mathbb{R}^5$, then $\dim(V \cap W) = 2$.

(14) Suppose A is a 2×2 matrix over real numbers with eigenvalues i and $-i$. Then

(A) A cannot be orthogonal.

(B) A cannot be symmetric.

(C) A cannot be skew-symmetric.

(D) A cannot be invertible.

(15) Let G be the group $G = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then

(A) G is isomorphic to S_3.

(B) G is isomorphic to a subgroup of S_4.

(C) G is isomorphic to a proper subgroup of S_5.

(D) G is not isomorphic to a subgroup of S_n for all $n \geq 3$.

(16) Let G be a group of order 121. Then

(A) G must be cyclic.

(B) G must have an element of order 11.

(C) G must have an element of order 121.

(D) G cannot have an element of order 11.

(17) For which of the following values of n does there exist a field of order n.

(A) $n = 6$.

(B) $n = 81$.

(C) $n = 21$.

(D) $n = 36$.

(18) The number of group homomorphisms $\varphi : \mathbb{Z} \to \mathbb{Z}$ is

(A) one

(B) two

(C) three

(D) infinity

(19) The set $[0,1] \times (0,1) \subset \mathbb{R}^2$ is

(A) open

(B) closed

(C) compact

(D) connected

(20) Which of the following sets is homeomorphic to

$$D = \{z \in \mathbb{C} : |z| \leq 1\}$$

(A) $\{z \in \mathbb{C} : |z| < 2\}$.

(B) $[0,1] \times (0,1)$.

(C) $\{z \in \mathbb{C} : |z| \leq 2, Re(z) \leq 1\}$.

(D) $(0,1) \times (0,1)$.

PART B

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and a a constant with $0 < a < 1$ so that $0 < f'(x) < a$ for all $x \in \mathbb{R}$. Define the sequence $\{x_n\}_{n\geq 0}$ by $x_0 = 0$ and $x_n = f(x_{n-1})$ for $n \geq 1$. Show that $|x_{n+1} - x_n| < a|x_n - x_{n-1}|$ for $n \geq 1$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and a a constant with $0 < a < 1$. Suppose the the sequence $\{x_n\}_{n\geq 0}$ defined by $x_0 = 0$ and $x_n = f(x_{n-1})$ for $n \geq 1$ satisfies $|x_{n+1} - x_n| < a|x_n - x_{n-1}|$ for $n \geq 1$. Show that x_n converges and that $x = \lim_{n\to\infty} x_n$ satisfies $f(x) = x$.

(3) Let $f(z)$ be a complex analytic function on $\mathbb{C} \setminus S$, where $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Suppose that there is an integer $k \geq 1$ such that

$$|f(z)| \leq |z|^k$$

for all $z \in \mathbb{C} \setminus S$. Show that all the singularities of f are removable.

(4) Let $f(z)$ be a complex analytic function of $\mathbb{C}$ satisfying, for some integer k,

$$|f(z)| \leq |z|^k$$

for all $z \in \mathbb{C}$. Show that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cz^k$.

(5) Let V and W be subspaces of $\mathbb{R}^n$ with $\dim(V) = \dim(W)$. Show that there is an isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(V) = W$. Here $T(V) = \{T(v) : v \in V\}$.

(6) Let V and W be subspaces of $\mathbb{R}^n$. Show that there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(V) = W$ if and only if $dim(V) \geq dim(W)$.

(7) Let G be a cyclic group such that G has exactly three subgroups, $\{1\}$, G and a proper subgroup H. Show that the order of G is p^2 for some prime p.

(8) Let G be a finite group such that G has exactly three subgroups, $\{1\}$, G and a proper subgroup H. Show that G is cyclic.

End of question paper