# INDIAN INSTITUTE OF SCIENCE
# BANGALORE - 560012

## ENTRANCE TEST FOR ADMISSIONS - 2010

**Program : Research**
**Entrance Paper : Mathematics**
**Paper Code : MA**

Day & Date
**SUNDAY, 25TH APRIL 2010**

Time
**9.00 A.M. TO 12.00 NOON**

# Instructions

(1) This question paper consists of two parts: Part A and Part B, and carries a total of 100 Marks.

(2) There is no negative marking.

(3) Candidates are asked to fill in the required fields on the sheet attached to the answer book.

(4) Part A carries 20 multiple choice questions carrying 2 marks each. Answer all questions in Part A.

(5) Answers to Part A are to be marked in the OMR sheet provided.

(6) For each question, darken the appropriate bubble to indicate your answer.

(7) Use only HB pencils for bubbling answers.

(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

(10) Part B has 8 questions. Answer any 6 in this part. Each question in this part carries 10 marks.

(11) Answers to Part B are to be written in the separate answer book provided.

(12) **Answer to each question in Part B should begin on a new page.**

(13) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ ($\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $\mathbb{C}_+$) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.

(14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denotes the standard norm on $\mathbb{R}^n$. The metric given by $d(x, y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

# MATHEMATICS

## PART A

(1) Let $\{x_n\}$ be an unbounded sequence of non-zero real numbers. Then,

(A) $\{x_n\}$ must have a convergent subsequence.

(B) $\{x_n\}$ cannot have a convergent subsequence.

(C) $\{1/x_n\}$ must have a convergent subsequence.

(D) $\{1/x_n\}$ cannot have a convergent subsequence.

(2) Let $f_n \colon [0,1] \to \mathbb{R}$ be the sequence of functions

$$f_n(x) = \begin{cases} \sin(n\pi x) & \text{if } x \in [0, 1/n], \\ 0 & \text{if } x \in (1/n, 1]. \end{cases}$$

Then,

(A) The sequence $\{f_n\}$ does not converge pointwise.

(B) The sequence $\{f_n\}$ converges pointwise but the limit is not continuous.

(C) The sequence $\{f_n\}$ converges pointwise but not uniformly.

(D) The sequence $\{f_n\}$ converges uniformly.

(3) Suppose $f \colon [0,1] \to \mathbb{R}$ is a function satisfying $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in [0,1]$. Then,

(A) $f$ is necessarily continuous but need not be differentiable.

(B) $f$ may be strictly decreasing.

(C) $f$ is necessarily constant.

(D) no such function $f$ exists.

(4) The number of symmetric, positive definite $8 \times 8$ matrices having trace equal to 8 and determinant equal to 1 is

(A) 0.

(B) 1.

(C) greater than 1 but finite.

(D) infinite.

(5) Suppose $K \subset \mathbb{R}^2$ is a connected set such that for all points $x \in K$, $K \setminus \{x\}$ (the complement of $x$ in $K$) is not connected. Then,

(A) $K$ must be homeomorphic to an interval of $\mathbb{R}$.

(B) $K$ must have empty interior.

(C) $K$ must be open.

(D) $K$ must be closed.

(6) Let $S$ be a collection of pairwise disjoint open sets in the plane $\mathbb{R}^2$. Then,

(A) $S$ cannot be finite.

(B) $S$ cannot be countably infinite.

(C) $S$ cannot be uncountably infinite.

(D) $S$ must be empty.

(7) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Then,

(A) $T$ must be continuous but is not necessarily uniformly continuous.

(B) $T$ must be uniformly continuous.

(C) $T$ is continuous if and only if $T$ is onto.

(D) $T$ is uniformly continuous if and only if $T$ is onto.

(8) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $\langle Tx, x\rangle = 0$ for all $x \in \mathbb{R}^n$. Then, it is necessarily true that

(A) $\mathrm{trace}(T) = 0$.

(B) $\det(T) = 0$.

(C) all eigenvalues of $T$ are real.

(D) $T = 0$.

(9) Let $G \subset (\mathbb{C}^*, \cdot)$ be a finite subgroup of the group $\mathbb{C}^*$ of non-zero complex numbers with multiplication as the group operation. Then we must have

(A) $\sum_{z\in G} z = 0$.

(B) $\sum_{z\in G} z = 1$.

(C) $\prod_{z\in G} z = 0$.

(D) $\prod_{z\in G} z = 1$.

(10) Suppose $A$ is a $2 \times 2$ matrix over real numbers with $\text{trace}(A) = 0$ and $\det(A) = 2$. Then $A$ may be

(A) orthogonal.

(B) symmetric.

(C) skew-symmetric.

(D) diagonal.

(11) Let $\{x_n\}$ be a sequence of real numbers so that $\sum_{n=1}^{\infty} |x_n - x| = c$, with $c$ finite. Then

(A) $\{x_n\}$ may not be bounded.

(B) $\{x_n\}$ must converge to $x$.

(C) $\{x_n\}$ must converge to $x + c$.

(D) $\{x_n\}$ is bounded but not necessarily convergent.

(12) For which of the following values of $n$ is every abelian group of order $n$ cyclic?

(A) $n = 12$.

(B) $n = 45$.

(C) $n = 8$.

(D) $n = 21$.

(13) Assume $a > 1$. Then, $\lim_{n \to \infty} n^{-2} e^{(\log(n))^a}$ is

(A) 0 for all $a > 1$.

(B) 0 if and only if $1 < a < 2$.

(C) $\infty$ for all $a > 1$.

(D) $\infty$ if and only if $1 < a < 2$.

(14) Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $x < f(x) \leq 1/2$ if $0 \leq x < 1/2$ and $1/2 \leq f(x) < x$ if $1/2 < x \leq 1$. Let $a \in [0,1]$ and define $x_n$ inductively by $x_1 = a$ and $x_{n+1} = f(x_n)$ for $n \geq 1$. Then $\lim_{n \to \infty} x_n$ is

(A) $1/2$ for all $a \in [0,1]$.

(B) 0 if and only if $0 \leq a < 1/2$.

(C) 0 if and only if $1/2 < a \leq 1$.

(D) 1 if and only if $1/2 < a \leq 1$.

(15) Let $f(x) = x^4 + ax^2 + bx + c$, where $a$, $b$, and $c$ are real numbers. Then as a polynomial over $\mathbb{R}$,

(A) $f(x)$ is irreducible if and only if $b^2 - 4ac > 0$.

(B) $f(x)$ is irreducible if and only if $b^2 - 4ac < 0$.

(C) $f(x)$ is always irreducible.

(D) $f(x)$ is always reducible.

(16) Consider the permutation group $S_6$ on 6 letters and let $H \subset S_6$ be a subgroup with 9 elements. It is necessarily true that

(A) $H$ is abelian but not cyclic.

(B) $H$ is cyclic.

(C) $H$ is not abelian.

(D) if $H$ is abelian then $H$ is cyclic.

(17) Consider a set $S$ of unit vectors in $\mathbb{R}^2$ such that $\langle x, y \rangle = -1/2$ if $x, y \in S$, $x \neq y$. Then, it is necessarily true that

(A) the set $S$ is linearly independent.

(B) the set $S$ generates $\mathbb{R}^2$.

(C) the set $S$ is either linearly independent or generates $\mathbb{R}^2$.

(D) if the set $S$ is linearly independent, then $S$ generates $\mathbb{R}^2$.

(18) The radius of convergence of $\sum_{n=0}^{\infty} z^{n!}$ is

(A) 0.

(B) 1.

(C) 2.

(D) $\infty$.

(19) The function $f(z) = e^{e^{1/z}}$

(A) is analytic at $z = 0$.

(B) has a removable singularity at $z = 0$.

(C) has a pole at $z = 0$.

(D) has an essential singularity at $z = 0$.

(20) Let $f: \mathbb{C} \to \mathbb{C}$ be a function. Then the region $\Omega = \{z \in \mathbb{C} : |e^{-f(z)}| < 2\}$ can be described as

(A) $\Omega = \{z \in \mathbb{C} : \mathrm{Re} f(z) > -\log(2)\}$.

(B) $\Omega = \{z \in \mathbb{C} : \mathrm{Re} f(z) < -\log(2)\}$.

(C) $\Omega = \{z \in \mathbb{C} : \mathrm{Im} f(z) > -\log(2)\}$.

(D) $\Omega = \{z \in \mathbb{C} : \mathrm{Im} f(z) < -\log(2)\}$.

# PART B

(1) Let $f, g\colon [0,1] \to \mathbb{R}$ be non-negative continuous functions such that

$$\sup_{0 \le x \le 1} f(x) = \sup_{0 \le x \le 1} g(x).$$

Show that $f(t) = g(t)$ for some $t \in [0,1]$.

(2) Let $f\colon [0,1] \to [0,1]$ be a function. Assume that, for every sequence $\{x_n\}$ in $[0,1]$, whenever both the sequences $\{x_n\}$ and $\{f(x_n)\}$ converge, we have

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Show that $f$ is continuous.

(3) Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$f'(x) = \frac{1}{x^2 + \sin^2(x) + f(x)}, \quad \forall x \ge 1,$$

and

$$f(x) \ge 0, \ \forall x \ge 1.$$

Show that $\lim_{x\to\infty} f'(x) = 0$. Deduce that $\lim_{x\to\infty} f(x)$ exists.

(4) Let $f(x)$ be a continuous function on $[0,1]$ satisfying

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 0.$$

Show that there exist $a, b \in [0,1]$, $a < b$, such that $f(a) = f(b) = 0$.

(5) Let $V$ and $W$ be vector spaces over $\mathbb{R}$ and $A\colon V \to W$ a linear transformation. Suppose there exists a unique $B\colon W \to V$ with $BA = I$, show that $AB = I$.

(6) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a surjective homomorphism from the additive group of integers to itself. Show that $f$ must be injective.

(7) Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be an injective homomorphism from the additive group of rationals to itself. Show that $f$ must be surjective.

(8) Let $p(z)$ and $q(z)$ be relatively prime polynomials with complex coefficients so that $\deg(q(z)) \ge \deg(p(z)) + 2$ and let $f(z) = p(z)/q(z)$. Show that the sum of the residues of $f(z)$ over all poles is 0.