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ENTRANCE EXAMINATION, 2014

Pre-Ph.D./Ph.D.
Mathematical Sciences

[Field of Study Code : MATP (160)]

Time Allowed : 3 hours

Maximum Marks : 70

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) The answer must be written in the space provided in the answer table for Section—A and in the space given after each question in the remaining Sections. Answer written in any other place will not be evaluated.

(iii) For each question in Section—A, *one and only one* of the four choices [(a), (b), (c), (d)] given is the correct answer. Each correct answer will be awarded +3 marks. Each wrong answer will be given −1 mark. If a question is not attempted, then no marks will be awarded for it.

(iv) Questions in Section—B carry 3 marks each. Questions in Section—C carry 5 marks each. Question in Section—D carries 6 marks.

(v) Answers to all the questions in Sections B, C and D must be **justified with mathematical reasoning**, or else they will be considered **invalid**.

(vi) In the following $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. For $A, B \subset \mathbb{R}$, $A \setminus B := \{x \in A \mid x \notin B\}$. Subsets of $\mathbb{R}^n$ are assumed to have the usual topology unless mentioned otherwise.

(vii) By $[a, b] \subset \mathbb{R}$ we denote the interval of all the real numbers between a and b including the end points. In order to exclude the end points $\{a, b\}$ from the interval, the notation (a, b) is used.

(viii) Extra pages are attached at the end of the question paper for rough work.

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SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO. | | | | | |

CENTRE OF EXAMINATION . . . .

. . . .

DATE. . . . .

....
(Signature of Candidate)

....
(Signature of Invigilator)

....
(Signature and Seal of Presiding Officer)

Not to be filled in by the candidate

Total of Section—A	
Total of Section—B	
Total of Section—C	
Total of Section—D	
Grand Total	

Answer table for Section—A

Question No.	Answer	Question No.	Answer
1.		**8.**	
2.		**9.**	
3.		**10.**	
4.		**11.**	
5.		**12.**	
6.		**13.**	
7.		**14.**	

SECTION—A

1. Let $A \in M_2(\mathbb{R})$ be such that it commutes with every 2×2 matrix with real entries. Then which of the following is necessarily true?

(a) A has distinct eigenvalues

(b) $\text{tr}(A) = 0$

(c) A is not invertible

(d) A is invertible if and only if it has at least one non-zero eigenvalue

2. Let $X = \{1, \ldots, n\}$ be given a topology in which there exist subsets of the following types :

$A \subset X$ is open but not closed, $B \subset X$ is closed but not open and $C \subset X$ is neither open nor closed.

Then the value of n must be

(a) at least 3

(b) at least 4

(c) exactly 6

(d) at most 7

3. Let G be a finite group such that $x^n = 1$ for all x in G and for a fixed $n \in \mathbb{N}$. Then

(a) n divides the order of G

(b) the order of G divides n

(c) the order of G and n are relatively prime

(d) None of the above holds

4. Consider the following Banach spaces over $\mathbb{C}$:

$X_1 := (\mathbb{C}^n, \|\cdot\|_\infty)$ $\quad$ $X_2 := l^1$

$X_3 := C([0, 1])$ with sup-norm $\quad$ $X_4 := (\mathbb{C}^n, \|\cdot\|_2)$

Which of the above is/are locally compact?

(a) Only X_1

(b) Only X_2 and X_3

(c) Only X_1 and X_4

(d) Only X_4

5. Consider the sets

$$A := \{\pi q : q \in \mathbb{Q}\},\ B := \mathbb{R} \setminus \mathbb{Z},\ C := \mathbb{R} \setminus \mathbb{Q} \text{ and } D := \bigcap_{x \in (0,1)} (\mathbb{R} \setminus \{x\})$$

Which of the above are dense in $\mathbb{R}$?

(a) All of A, B, C and D

(b) Only A, B and C

(c) Only C and D

(d) Only B and C

6. Let $X = [0, 1] \cup \{5, 6, 7, 8, 9, 10\}$ and let $f : X \to \mathbb{R}$ be a continuous function. Which of the following is ***not*** necessarily correct?

(a) f is uniformly continuous

(b) f is differentiable

(c) f is bounded

(d) The image of f is closed

7. Let l^2 be the Hilbert space over $\mathbb{C}$ (of square summable sequences). Let $T : l^2 \to l^2$ be a bounded linear operator. Which of the following is necessarily true?

(a) For a $\lambda \in \mathbb{C}$, if $|\lambda| > \|T\|$, then λ does not belong to the spectrum of T

(b) $\sum_{i=1}^{\infty} \langle Te_i, e_i \rangle$ converges, where $\{e_i\}_{i \in \mathbb{N}}$ is the standard orthonormal basis

(c) If T is injective, then it is surjective

(d) If T is surjective, then it is injective

8. Let K and F be field extensions, both of degree 3 over $\mathbb{Q}$. Let $L = K \cdot F$ be the composite, that is, the smallest field extension of $\mathbb{Q}$ containing both K and F. Then the extension degree $[L : \mathbb{Q}]$ could be

(a) 3 or 9, but neither 5 nor 6

(b) 9, but none of 3, 5 or 6

(c) 3 or 6 or 9, but not 5

(d) any of 3, 5, 6 and 9

9. Let X be a complete metric space and $A \subset X$. Which of the following conditions implies that A is compact?

(a) A is closed and bounded

(b) A is totally bounded

(c) A is closed and totally bounded

(d) There exists a continuous map $f : X \to \mathbb{R}$ such that $f(A)$ is compact

10. Consider $W_1 = \{(a, b, a, c) : a, b, c \in \mathbb{C}\}$ and $W_2 = \{(a, 0, -a, 0) : a \in \mathbb{C}\}$ as subspaces of $\mathbb{C}^4$ over $\mathbb{C}$. Then

(a) $\dim (W_1 + W_2) = 1$

(b) $\dim (W_1 + W_2) = 2$

(c) $\dim (W_1 + W_2) = 3$

(d) $\dim (W_1 + W_2) = 4$

11. Let R be a ring with 1 and let $f(x) \in R[x]$ be a polynomial of degree n. Then the number of roots of f in R

(a) equals n

(b) is at most n

(c) is infinite if R is infinite

(d) cannot be determined in general

12. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a surjective linear map. Consider the following statements :

[S1] The map f is one-one.

[S2] The map f is continuous.

[S3] The map f is differentiable.

Which of the above is/are necessarily true?

(a) All of [S1], [S2] and [S3]

(b) [S2] and [S3], but not [S1]

(c) Only [S1]

(d) Only [S2]

13. Let $A \in M_n(\mathbb{C})$ $(n \geq 3)$ be a matrix which has one eigenvalue of multiplicity at least two. Consider the following statements :

[S1] All eigenvalues of A are real.

[S2] All eigenvalues of A are equal.

[S3] A is not invertible.

[S4] A is not diagonalizable.

Which of the above statement(s) is/are necessarily true?

(a) All

(b) Only [S3]

(c) None

(d) Only [S3] and [S4]

14. Let X be a set. Consider the following statements :

[S1] If there is a surjective map $f : X \to \mathbb{N}$, then X is countable.

[S2] If there is an injective map $f : X \to \mathbb{N}$, then X is countable.

[S3] If there is a surjective map $f : X \to \mathbb{Q}$, then X is countable.

[S4] If there is a surjective map $f : \mathbb{Q} \to X$, then X is countable.

Which of the above statements are correct?

(a) [S1], [S2]

(b) [S2], [S4]

(c) [S3], [S4]

(d) [S1], [S4]

SECTION—B

15. Let Ω be an open connected subset of $\mathbb{C}$ with $i \in \Omega$ and let $f : \Omega \to \mathbb{R}$ be holomorphic (complex analytic) such that $f(i) = -1$. Calculate the derivative of f at i, (where $i = \sqrt{-1}$)

16. Let X be a finite set with 10 elements and let a be an element of X. Then how many subsets of X contain a?

17. Let $X \subset \mathbb{R}$ be of positive finite Lebesgue measure. Prove or disprove the following statement :

There exists an open interval $(a, b) \subset X$ for some $a, b \in \mathbb{R}$, $a < b$.

18. Find all finite subgroups of $(\mathbb{C}, +)$. Give an example of an infinite proper subgroup of $(\mathbb{C}, +)$.

SECTION—C

19. Prove that any finitely generated subgroup of $(\mathbb{Q}, +)$ is not dense in $\mathbb{R}$.

20. Prove or disprove the following statements :

(i) There is a surjective continuous map from $[0, 1]$ to $(0, 1)$.

(ii) There is a surjective continuous map from $(0, 1)$ to $[0, 1]$.

SECTION—D

21. Let $C(\mathbb{R})$ be the ring of continuous real-valued functions on $\mathbb{R}$. Consider the subsets I and J of $C(\mathbb{R})$, where I consists of compactly supported functions and J consists of functions vanishing at ∞, respectively. Prove or disprove the following statements :

(i) I is a prime ideal.

(ii) J is an ideal.